

Chapter 4

The Evolution of the Orbital Motion of Mars and Its Environment

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Abstract

By solving differential equations for the interaction of Solar-system bodies using the Galactica system, the evolution of the orbits of Mars, its satellites Phobos and Deimos, and three minor planets, Ceres, Pallas, and Vesta, is analyzed. The initial conditions and the solution method are discussed, and then the results are considered. The evolution of the Mars orbit is presented over several time intervals lasting 3 to 100 million years. The orbits of the minor planets are considered over two intervals of shorter duration. The evolution of the orbits of Mars' satellites is presented over a greater number of intervals. The periods and amplitudes of oscillations of orbital parameters are found. The results are confirmed by observations. Relations approximating the variation of parameters are obtained. The orbits evolve as a result of four motions. The shape of the orbit, i.e., its eccentricity, varies in time, and the orbit rotates in its plane, i.e. the perihelion of the orbit rotates. The orbital axis, i.e., the perpendicular to the orbits plane, precesses and oscillates relative to the angular momentum of the Solar system. The satellites' orbital axes precess and oscillate relative to the Mars orbital axis.

Keywords: N-body interaction, orbits, evolution, Mars, Ceres, Pallas, Vesta, Phobos, Deimos.

1. Introduction

The evolution of the orbital motion of Solar-system bodies can be established by solving a system of differential equations for gravitational interaction of N bodies. In vector form, this system can be written as follows [1]:

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$$\frac{d^2 \vec{r}_i}{dt^2} = -G \sum_{k \neq i}^n \frac{m_k \vec{r}_{ik}}{r_{ik}^3}, i = 1, 2, \dots, n, \quad (1)$$

where $\vec{r}_i$ is the radius-vector of the body with mass m_i relative to the center of mass of the entire n-body system;

$G = 6.67259 \times 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}$ is the gravitational constant;

$\vec{r}_{ik}$ is the radius-vector drawn from the body with mass m_k to the body with mass m_i .

In the 18th and 19th centuries, the theory of secular perturbations was developed [2, 3] to treat the interaction of two bodies: the Sun and a planet, with the action of the remaining bodies being considered as perturbations. Those perturbations were represented in the form of series. Thus, a theory was created for the evolution of six orbital elements. This approach offers an approximate analytical method for solving equations (1). Nowadays, the series used contain hundreds of terms, and problems concerning the orbital evolution of Solar-system bodies are solved using computers [4–6]. When expanding expressions into series, a number of restrictions are imposed on the ranges of parameters; those restrictions are not always properly observed. There are also some other theoretical simplifications applied that introduce differences from the exact equations. Therefore, deviations of solutions obtained in this way over large time intervals from possible exact solutions are inevitable.

In the second half of the 20th century, in connection with the development of computers, numerical methods came into use for solving unsimplified equations (1) [7, 8]. Numerical solutions also involve errors but, over time, computing methods and computers themselves had evolved, and errors decreased. We also worked in this field [1, 9].

In addition to the primary gravitational interaction, various authors included into consideration additional minor effects, such as asteroid rings, a non-spherical shape of bodies, relativistic corrections, light effects, and so on. However, the constants of these minor effects are either unknown or questionable, and some of these additions lack sufficient physical substantiation. Furthermore, almost all of added factors violate the fundamental laws of mechanics. Therefore, such additions can only be used as tests for various hypotheses [10].

In this regard, we believe that the primary objective is to solve the problem of evolution of Solar-system bodies (1) without simplification yet with the use of the most accurate numerical method possible. The resulting evolution thus obtained will serve a benchmark against which the influence of additional minor effects can be considered.

Interest in studying the evolution of the orbital motion primarily stems from long-term climate variations. The Astronomical theory of climate changes involves a number of problems, of which the evolution of the planet's

orbit is the first one [11]. We previously solved the problem on the evolution of Mars' orbit over 100 million years [8,12], considering the interactions between the Sun, the planets, and the Moon. However, the magnitude of the influence on Mars of the nearest minor planets – Ceres, Pallas, and Vesta – as well as the moons of Mars, Phobos and Deimos, still remains unclear. Could this influence lead to significant changes in the Mars climate over a long time interval as a result of these bodies' interactions? Therefore, a new approach was undertaken to tackle this problem, taking into account the influence of these bodies.

First-type initial conditions for the Galactica system [1,9] which were used to solve the problem of orbital evolution over the last 100 million years and the period of 2.7 million years into the future, were specified in the sn49f.dat file. In this file, the coordinates and velocities of bodies were adopted for the epoch of 12.30.1949 with the Julian day $JD_0 = 2433280.5$ in the equatorial barycentric coordinate system of 1950.0 with the Julian day $JD_s = 2433282.4234$. Modified DE19 ephemeris data obtained by NASA's Jet Propulsion Laboratory (JPL) were used here.

Later, the file with second-type initial conditions Sn49JPLc.dat was used for various calculations of the orbital motion and for solving the problem of evolution of the rotational motion of Earth [13]. In that file, the coordinates and velocities of bodies for the epoch of 12.30.1949 in the equatorial barycentric coordinate system of 2000.0 with the Julian day $JD_s = 2451545$ were used. They were determined from the DE406/LE406 ephemeris [14,15]. With the initial conditions of the second type, the problem of orbital evolution was solved for the period of the last 20 million years and over the period of million years into the future [11]. For the initial conditions of the first type, the masses, the coordinates, and the velocities of bodies are given in [8], and for the initial conditions of the second type, in [16].

In JPL theories, in addition to the Newtonian gravitational interaction of point bodies, a number of other minor effects were added. The solution to this problem is expressed as a series involving hundreds of coefficients. The values of those coefficients are refined by comparing the solutions with the entire set of observational data. Therefore, the resulting approximation system is quite good and agrees well with observational data.

Over time, the observational base used cannot ensure comparability of the results of this system with new data, so NASA releases new DE ephemerides at intervals of several years.

It should be noted that the Galactica system uses no additive weak effects. As already noted, many of such effects are inspired by various physical theories and are not sufficiently substantiated. Other effects lack precise interaction constants. Therefore, such effects degrade results of the main Newtonian gravitational interaction. The Galactica system uses a high-precision method for integrating differential equations (1) [1,9]. It is

orders of magnitude more accurate than the method used in DE ephemerides [17]. Therefore, the Galactica system made it possible to solve the orbital problem with first-type initial conditions over a period of 100 million years with acceptable accuracy [8, 12].

Third-type initial conditions were specified based on one of the latest NASA ephemerides, DE422, at the initial epoch of 2000.0. Since new ephemerides could lose accuracy over subsequent epochs, we have compared different ephemerides – DE200, DE403, DE405, DE406, and DE422 – at different epochs. It turned out that the best matching exists between DE406 and DE422 for the epoch of March 2, 1990, with $JD = 2447952.5$. Therefore, the coordinates and velocities of the planets, the Sun, and the Moon for this epoch were borrowed from the DE422 ephemeris, and those of the minor planets – Ceres, Pallas, and Vesta, from the NASA website <https://ssd.jpl.nasa.gov/>. Using these data, the initial-conditions file Sn90D422.dat, containing data for $k_2 = 14$ bodies, was compiled for the Galactica system: the Sun, the nine planets, the Moon, and the three minor planets. In addition, the file Sn49D422.dat was created for December 30, 1949, and the file Sn69D422.dat for June 28, 1969, respectively.

2. Algorithm and programs for setting the initial conditions

2.1. Setting algorithm

In the above-mentioned files, the velocities corresponded to the system of differential equations used in JPL theories. For correcting the velocities of bodies for the system of equations used in the Galactica program, a correction algorithm was developed, implemented in the CrVelF01 correction program.

For correcting the velocities contained in the file, another epoch is to be taken from the initial epoch (Initial), to be considered as a reference epoch (Etalon). The Galactica program calculates the motion of bodies from the initial to reference epoch. The resulting coordinates, for example, the coordinate of the k -th body $x_{g,k}$, are compared with the coordinate of this body in the reference file $x_{et,k}$, and their difference $\Delta x_{g,k} = x_{g,k} - x_{et,k}$ are determined. Using all the three coordinates, the distance r_k of the k -th body from the origin, the error Δr_k , and the relative error $\delta r_k = \Delta r_k / r_k$ were determined. Then, the average error for all the k_2 bodies was calculated:

$$\delta r_m = \sum_{k=1}^{k_2} \delta r_k / k_2. \quad (2)$$

The velocities of bodies in the initial epoch were refined as a result of successive iterations, and the average error δr_m served an efficiency indicator of the iterative process.

We denote the x -projection of the velocity at the i -th iteration as $v_{k,x,i}$, and the error δr_k at this velocity as $\delta r_{k,i}$. Then, at the previous iteration, those quantities have values $v_{k,x,i-1}$ and $\delta r_{k,i-1}$, respectively. Then, at the next iteration, the velocity in the initial epoch can be determined by linear iteration as follows:

$$v_{k,x,i+1} = v_{k,x,i} - k_{o,k,x} \cdot \delta r_{k,i} (v_{k,x,i} v_{k,x,i-1}) / (\delta r_{k,i} - \delta r_{k,i-1}), \quad (3)$$

where $k_{o,k,x}$ is the coefficient of attenuation of the influence of error $\delta r_{k,i}$ on the x -component of the velocity of the k -th body.

The magnitude of the error $\delta r_{k,i}$ is defined by the three velocity components of the k -th body. Therefore, the estimate $k_{o,k,x} \leq 1/3$ can be used to initially determine the value of $k_{o,k,x}$. We can subsequently refine this value based on the results of an iterative process.

2.2. Correction

The CrVelF01.exe executable program works with the following files: Initial.dat, Etalon.dat, CrVelD1.dat, VelDr.dat, Start.dat, MinfilFn.dat, StMin.dat, Endfile, and CrVelB1.bat. Initial.dat is the initial-conditions file for the Galactica system, with the velocities to be refined using the coordinates of the initial-conditions file Etalon.dat at a different epoch.

The CrVelD1.dat file contains all parameters of the iterative process, and the current and final results of this process are to be written into that file. Ten numbers are specified at the beginning of the file: the iterative-process number I , the iteration record-line number I02, the total number of iterations N , the total number of bodies k_2 , the iterative-process start date DATA, the absolute value of the permissible relative velocity increment dUmlma when the velocity is refined using formula (3), the iteration number Imin with the minimum difference drlm, its value drmin, the average error (2) at the penultimate step drlp and at the final step drlm. The first six values must be specified when starting the iterative process.

Next, the CrVelD1.dat file contains k_2 lines of attenuation coefficients $k_{o,k,x}, k_{o,k,y}, k_{o,k,z}$. Those lines are also specified at the beginning of the iteration process and can be modified during that process. Next, there follow 100 lines containing the results of each iteration process. Those results include the iteration process number I , the iteration record-line number I02, the number of the body $k_{ma,i}$ with the maximum error dr_k , the magnitude of this error drlmai, and the average error drl2mi at this iteration.

If the number of iterations N is greater than 100, then subsequent iterations are to be written from the first line. Therefore, with a large number of iterations, those 100 lines contain information about all the last 100 iterations.

The CrVelD1.dat file ends with k_2 lines containing the body numbers k and the relative error dr_k of each body at the previous and last steps.

The VelDr.dat file contains k_2 lines of velocity projections $\nu_{k,x,i-1}$, $\nu_{k,y,i-1}$ and $\nu_{k,z,i-1}$, and the error $dr_{k,i-1}$ at the $i - 1$ -th iteration. The values of the same quantities then follow at the i -th iteration. This file must be generated before starting the iteration process. The errors $dr_{k,i-1}$ for the last iteration step can be omitted.

The Start.dat file is generated by the CrVelF01.exe program based on the corrected velocities according to formula (3). The Galactica program begins calculating the motion of bodies from this file till the reference epoch. The calculation results are output in the gaze28.dat file, where the CrVelF01.exe program compares the coordinates of the bodies with those of the Etalon.dat file. Then, using formula (3), newly refined velocities are calculated.

The iteration process is uneven. Steps with the successively decreasing error δr_m are interrupted by steps in which the error does not decrease. Therefore, at each iteration step the minimum error δr_{min} is determined and the file MinfilFn.dat is generated. Its first line displays the iteration number I and the average relative difference δr_{min} . Then, there follow k_2 lines containing velocities $\nu_{k,x}$, $\nu_{k,y}$ and $\nu_{k,z}$, and the relative error δr_k due to those velocities.

In addition, at this same iteration the starting file StMin.dat is generated, which uses those velocities. After completing a total of N iteration cycles, the files MinfilFn.dat and StMin.dat containing the iteration data with minimal errors are saved.

The program CrVelB1.bat was created to control the CrVelF01 and Galactica programs. It runs those programs one by one and additionally performs a number of other actions. Upon completion of N iterations, the program CrVelF01 generates the file Endfile. The first line records the last-iteration number I , the total number of iterations N , and the last error δr_m . Then, in the k_2 lines the last errors δr_k for each body are listed.

The appearance of the Endfile file signals about the termination of the CrVelB1.bat control program. This program is also terminated if errors occur in the programs it controls or if necessary files are missing.

2.3. Initial-conditions files for the Galactica system

Using this CrVelF01 iterative system, the velocities in the Sn90D422.dat file were refined. This file was used as Initial.dat, and the Sn69D422.dat file for the epoch of June 28, 1969 was used as Etalon.dat. Using these two files, the Sn90A422.dat file with refined velocities was generated.

A similar correction of the Sn90D422.dat file was performed using the Sn49D422.dat file for the epoch of December 30, 1949, yielding the Sn90B422.dat file. Using these two files, the Sn90C422.dat file containing the half-sum of the velocities of the first two files was generated.

It should be noted that the experience gained when performing corrections

has shown that, using the correction by a single epoch, one can achieve very small, almost zero, errors. However, calculations up to another epoch will yield large errors. Finally, the file obtained by correction for two epochs yields similar errors for all other epochs.

For calculations up to 06.28.1969, the final file Sn90C422.dat yielded an average error δr_m that was 15.7 times smaller than the accuracy that could be achieved with the initial file Sn90D422.dat. For calculations up to 12.30.1949, 11.5 times smaller error could be achieved.

The final file Sn2tC422.dat of the third initial conditions for the Galactica system at the epoch of 2000.0 with the Julian day $JD_s = 2451545$ was derived from the file Sn90C422.dat for the epoch of 03.02.1990 by the calculation performed using the Galactica program up to the epoch of 2000.0.

The initial-conditions file Sn2tMa2S.dat with the Martian satellites Phobos and Deimos was obtained based on the file Sn2tC422.dat. The data for those satellites for the epoch of 2000.0 were taken from the NASA website. Their velocities were also corrected using the CrVelF01 system. Due to the short orbital periods of these satellites, 0.319 and 1.263 days, respectively, the velocities were corrected for epochs occurring ± 1 day before and after the epoch of 2000.0. The masses, the radii, the coordinates, and the velocities of bodies contained in this file are given in Table A1 of Appendix. The coordinates and velocities for the epoch of 2000.0 are given in the heliocentric equatorial coordinate system for the same epoch of 2000.0.

3. Some details of the study of the evolution of the orbital motion

3.1. Coordinate system

In the initial-conditions file, the coordinates and velocities of bodies are used in the barycentric xyz coordinate system, i.e., the origin O (Fig. 1) is the center of mass of the Solar system. By integrating the differential equations of motion of Solar-system bodies [1], the Galactica program determines the coordinates of bodies. When analyzing planetary orbits, the coordinates of the Sun and the planet are obtained for one of its revolutions, and for a satellite, its coordinates and the coordinates of its planet are obtained. Based on those data, the trajectory of the body is calculated, i.e., its orbit and orbital parameters in the equatorial coordinate system.

The Galactica system determines 12 orbital parameters, three of which are shown in Fig. 1. The angle characterizing the position of the ascending node γ_2 of the planet's orbital plane $PlPl$ at the epoch T is determined by the arc $\gamma_0\gamma_2$: $\varphi_\Omega = \gamma_0\gamma_2$. The inclination of the planets orbital plane ($PlPl$) to the fixed-equator plane A_0A_0 is shown with the angle i , and the angular distance

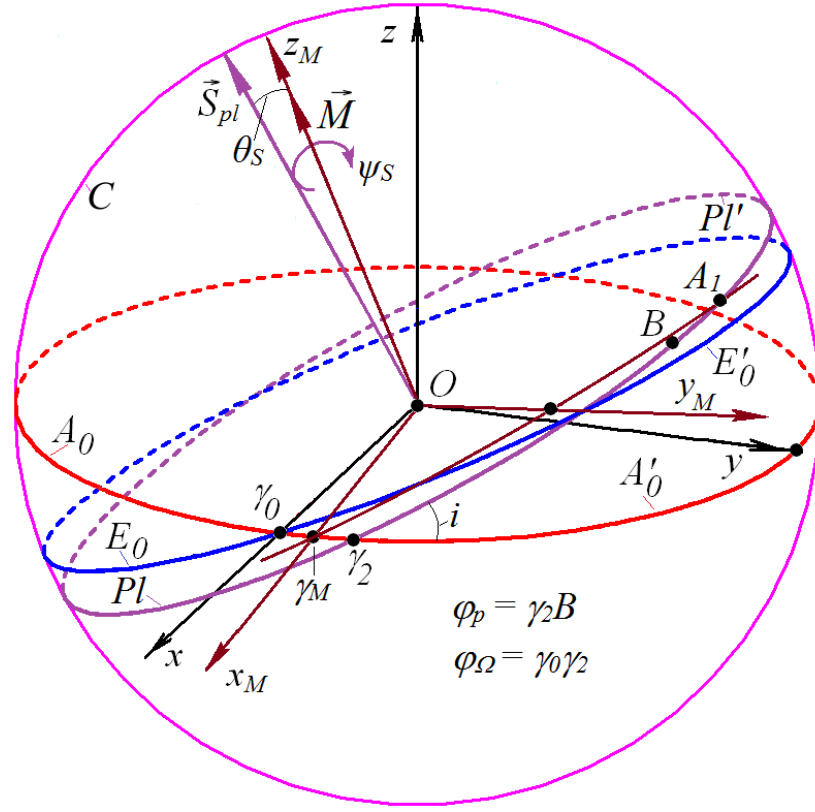


Figure 1. Parameters of a planet's orbit ($PlPl'$) in the fixed equatorial system xyz : C is the celestial sphere; fixed planes at the epoch of 2000.0, $JD_s = 2451545$: $A_0A'_0$ is the Earth equator, $E_0E'_0$ is the Earth orbit (plane of ecliptic); γ_0 is the vernal equinox at the epoch of 2000.0; $PlPl'$ is the moving plane of the planet's orbit (Mars) at the epoch T ; unit vectors: $\vec{S}_{Pl}$ is the unit vector of the axis of the planet's orbit, $\vec{M}$ is the unit vector of the angular momentum of the Solar system; B is the position of the planet's perihelion on the celestial sphere; $\varphi_\Omega = \gamma_0\gamma_2$ is the angular distance of the ascending node of the orbit; $\varphi_p = \gamma_2 B$ is the angular distance of the perihelion; i is the angle of inclination of the planet's orbital plane (Pl) to the plane of fixed equator ($A_0A'_0$); $\gamma_M A_1$ is the perpendicular to $\vec{M}$ plane (Laplace plane), γ_M is its ascending node; $\varphi_M = \gamma_0\gamma_M$; $i_M = \gamma_2\gamma_M A_1$ is the angle of inclination of the Laplace plane to the equatorial plane; $\psi_S = \gamma_M A_1$ is the angle of precession of the orbital plane; and $\theta_S = \gamma_2 A_1 \gamma_M$ is the angle of inclination of the orbital plane to the Laplace plane.

of the projection of the orbits perihelion B onto the celestial sphere C from the ascending node γ_2 is determined by the arc $\gamma_2 B$: $\varphi_p = \gamma_2 B$.

3.2. Comparison with observations

In astronomy, secular variations of planetary orbital parameters have been established based on the observational data gained over intervals of the order of a thousand years. Approximation relations for those variations were obtained by [18] and [19]. We determined the variations of planetary orbital parameters using the Galactica system over several thousand years based on the initial conditions of the first and second type. In Fig. 1 of the work by [16], those variations of Mars' orbital parameters over the time interval from -3.4 to 3.6 thousand years from the epoch of 1950 are presented for 6 orbital parameters: orbital eccentricity e , angles i , φ_Ω and φ_p , and the deviations from the mean values of the orbital semi-major axis Δa and the orbital period ΔP . Those parameters were evaluated using the initial conditions of the second type. The graphs show that, over this time interval, the eccentricity e and the perihelion angle φ_p both increase with time, while the longitude of the ascending node of the orbit φ_Ω decreases. The inclination angle i of the Mars orbit varies non-monotonically. In the modern epoch, at the time of $T = 1400$ years from December 30, 1949, this angle has a maximum value. In contrast to those parameters, the semi-major axis a and the orbital period P remain unchanged on average; therefore, the graphs show their deviations from the average values. These fluctuations are small in comparison with the average values $a_m = 2.2794 \times 10^8$ km and $P_m = 1.8808$ year. The parameters e , i , φ_Ω and φ_p also fluctuate with the same relative amplitudes as the parameters a and P . The periods of those fluctuations are small. For example, for the eccentricity e they are equal to 5.94 and 11.86 years.

As a result of those fluctuations, the deviations of the semi-axis from the mean value a_m range from -1.1×10^4 to 1.6×10^4 km. And the length of the year relative to the mean value P_m varies in the range from -0.85 hours to 1.2 hours. Such small fluctuations of the semi-axis and period also occur over larger time intervals. For example, in the interval of 10 million years considered below, the range of oscillations of the semi-axis a is from -1.4×10^4 to 2.0×10^4 km, and the range of oscillations of the period P is from -1.0 to 1.6 hours. Similar small fluctuations of those parameters relative to the mean values also occur for other planets.

The graphs in Fig. 1 [16] additionally show the approximation dependences obtained by [18] and [19]. The eccentricity e and the angles i , φ_Ω and φ_p coincide with observations over an interval of ± 1000 years, i.e., within the limits of the approximations. The calculations for the semi-axis a and the period P also coincide with observations, and the obtained oscillations are comparable in value to the difference between the approximations proposed

by different authors. Similar studies of secular variations of orbital parameters have been carried out for other planets. They are also compared with approximations of observational data, with the exception of Pluto. For Pluto, there is no approximation of observational data.

3.3. Mercury's perihelion, accuracy criteria, and other details

For all planets and all orbital parameters, the calculated secular variations agree with the approximations to observational data, with the exception of Mercury's perihelion. Observations indicate a 10% excess rotation of Mercury's perihelion [20]. Our research has established that the excess rotation of Mercury's perihelion is due to the oblateness of the Sun. We developed a compound model of Sun's rotation that takes its oblateness into account [21]. By solving the problem of the interaction of Solar-system bodies with this model of the Sun [22], we obtained changes in the planetary orbital parameters that proved to be in perfect agreement with the approximations of observational data [18, 19]. Therefore, we concluded that Newton's law of gravitation completely describes the interactions and motions of Solar-system bodies.

The Galactica program uses about 10 criteria for problem solution accuracy, which are available in the output file. Those criteria can be used for monitoring the solution accuracy. The most effective of these criteria is the relative change in the projection of the angular momentum of the Solar system δM_z onto the z -axis. For example, when solving the problem over 100 million years, we obtain the value $\delta M_z = 7.9 \times 10^{-11}$, and in the solutions considered below for 10 million years, the value $\delta M_z = 1.54 \times 10^{-11}$. Other authors, for example, [7] report changes of this value over the same time intervals by more than 4.5 orders [8]. For planets and, especially, for their satellites, when solving the problem over large time intervals, the solution accuracy can also be controlled considering the constancy of the average period of revolution P , or the semi-axis a .

Solving a problem over long periods of time takes a significant amount of computing time. For example, a 100-million-year problem with $n = 11$ bodies took two years to solve in 2002. Supercomputer performance has recently increased, and a 10-million-year problem with $n = 14$ bodies took 39 days to solve. According to a 2009 estimate [8], on a computer with the same clock frequency, one step in the Galactica algorithm is calculated twice as fast as in [7].

The Galactica system is universal and suitable for studying the dynamics and evolution of both planetary systems and stellar associations [23]. It is also suitable for studying the particle interactions according to Coulomb's law [24, 25]. The system is available in archived form at <https://www.ikz.ru/~smulski/GalactcW/GalacFoldrs.zip>, and its description is

available at <https://www.ikz.ru/~smulski/GalacticW/GalDis crE.pdf>.

The solution of all problems considered below was performed on supercomputers of the Central Shared-Use Center of the Siberian Supercomputer Center of the Institute of Computational Mathematics and Mathematical Geophysics, Novosibirsk.

4. Evolution of Mars' orbit

4.1. Results for the last 3 million years

In those studies, the differential motions of 14 bodies with the initial-conditions file Sn2tC422.dat, i.e. without the satellites of Mars, were integrated with a step $dT = 1 \times 10^{-4}$ year over the period of the last 10 million years. The results of the calculations of the evolution of the Mars orbit for the period of 3 million years into the past are presented in Fig. 2. The eccentricity e experiences short-period oscillations with an amplitude of 0.024 with the first period $T_{e1} = 95$ thousand years.

There also exist a second period of eccentricity oscillations $T_{e2} = 2.32$ million years with an amplitude of 0.032. The average value of the eccentricity is $e_m = 0.066$. The longitude of the ascending node φ_Ω varies with an average period $T_\Omega = -T_S = 73$ thousand years around the average value $\varphi_{\Omega m} = 0.0678$ radians, where T_S is the precession period of the Mars orbital axis $\vec{S}_{pl}$ (Fig. 1). In the future, the precession of orbital axes [17,26] will be considered in more detail.

The angle of inclination of the orbital plane i to the equatorial plane of 2000.0 oscillates with the same period $T_i = -T_S = 73$ thousand years around the average value $i_m = 0.405$ radians. Over an interval of 3 million years, the angle of perihelion φ_p , on the average, increases almost linearly with time, i.e., the perihelion moves in the direction of Mars' revolution around the Sun, completing, on the average, one revolution in $T_p = 77.5$ thousand years. This period T_p is calculated for a base of the time span of 10 million years. The linear approximation with this period is shown by the thin line. As evident from Fig. 2, there are deviations of this line from the ω_p graph. This is due to the fact that the perihelion rotates unevenly. Its angular velocity ω_p oscillates with an average period of 87 thousand years. Moreover, in the epoch $T = -1.355$ million years, it assumes a large negative value -4910 ''/century (arc seconds per century) with an average value $\omega_{pm} = 1672$ ''/century. In the epoch of inverse rotation of Mars' orbit perihelion, its eccentricity is $e = 9 \times 10^{-4}$, i.e., it is nearly zero, and the orbit is a circle.

The graphs of variation of Mars orbital parameters presented in Fig. 2 perfectly coincide with the graphs of previous calculations (see Fig. 2 in [16]). The latter solutions were based on the first-type initial conditions for the epoch

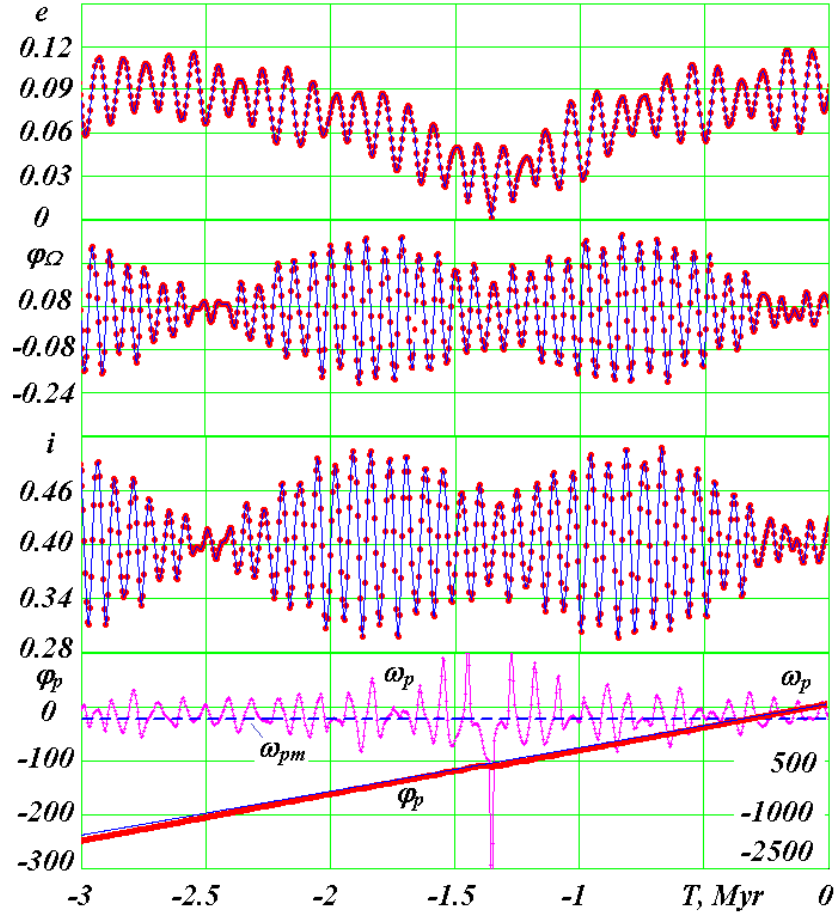


Figure 2. Evolution of the Mars orbit over the last 3 million years: e is the eccentricity; i is the inclination of the orbital plane to Earth's equatorial plane of 2000.0 in radians; φ_{Ω} is the angular position of the ascending node of the orbit from the x -axis at the epoch of 2000.0 in radians; φ_p is the angular position of the perihelion in the orbital plane from the ascending node in radians; T is the time in millions of years; the interval between adjacent points is 5 thousand years; ω_p is the angular velocity of perihelion rotation in $''/\text{century}$ over a time interval of 10 thousand years; $\omega_{pm} = 1672''/\text{century}$ is the average angular velocity of perihelion rotation; the thin lines in the graphs of i and φ_{Ω} show dependences (13) and (14).

of December 30, 1949, and the interaction of the Sun, the planets, and the Moon was considered. This coincidence indicates that the minor planets Ceres,

Pallas, and Vesta do not have a noticeable influence on the evolution of Mars' orbit. Second, the increased accuracy of the new initial conditions does not lead to noticeable changes in the evolution of the orbits of Solar-system bodies under consideration. Therefore, all results obtained with the previous initial conditions remain valid.

4.2. Results for the last 10 million years

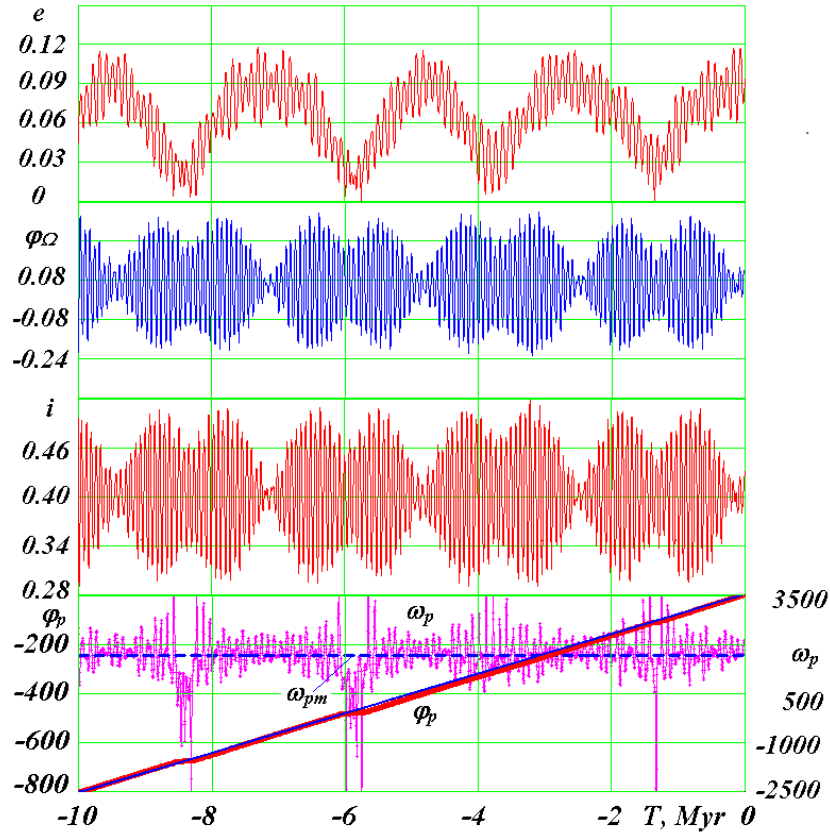


Figure 3. Evolution of Mars' orbit over the last 10 million years. For designations, see Fig. 2.

Figure 3 shows the evolution of Mars' orbital parameters over the last 10 million years. The eccentricity e fluctuates between 3.3×10^{-4} and 0.118, with an average value of $e_m = 0.066$. There are two periods of eccentricity oscillation: $T_{e1} = 95$ thousand years and $T_{e2} = 2.32$ million years, with average amplitudes of $e_{A1} = 2.39 \times 10^{-2}$ and $e_{A2} = 3.65 \times 10^{-2}$, respectively.

The main period of oscillation of the orbital inclination angle i to the equatorial plane of 2000.0 years is $T_i = T_S = -73$ thousand years, around the mean value of $i_m = 0.405$ radians. The position of the ascending node φ_Ω oscillates with the same period around the mean value of $\varphi_{\Omega m} = 0.068$ radians. This period is caused by the clockwise precession of the orbital axis $\vec{S}$, i.e., the precession counterclockwise to the Mars orbital motion. In addition, there are also oscillations of the orbital axis with several periods, which are manifested in the variations of the angles i and φ_Ω .

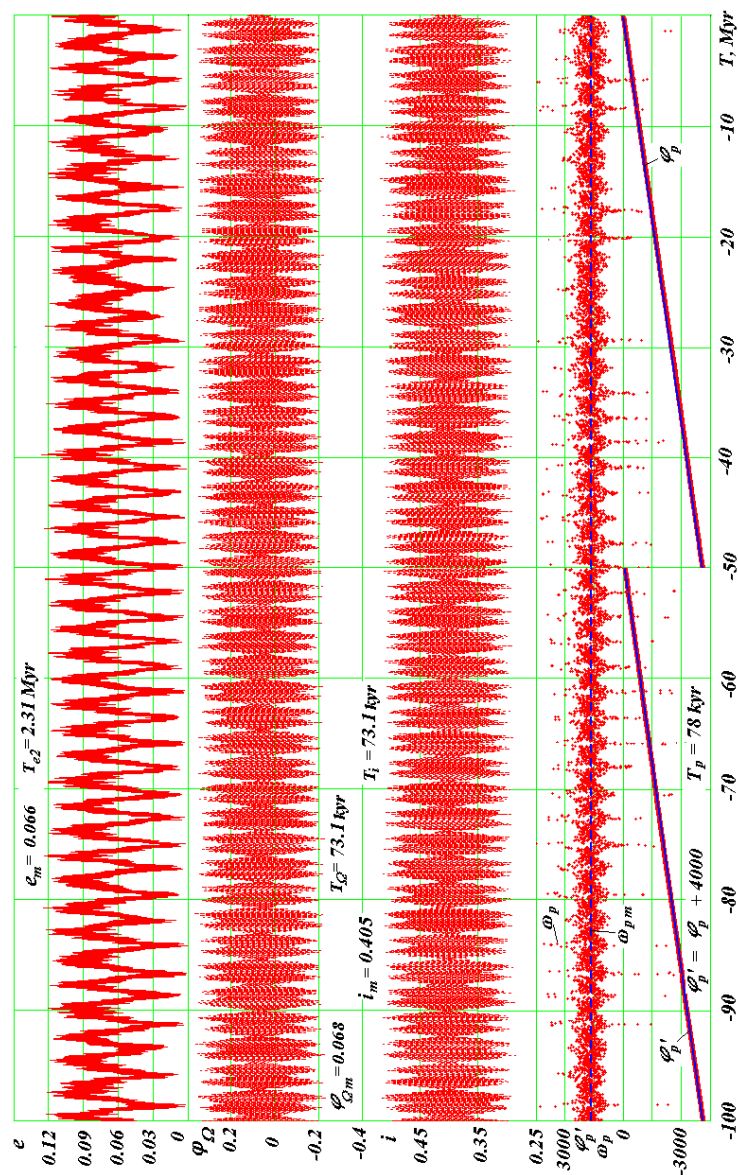
The average perihelion rotation period of Mars over 10 million years is $T_p = 77.5$ thousand years. The linear approximation of the perihelion angle φ_p with this period is shown by the thin line. It deviates from the φ_p graph because the perihelion rotates unevenly. Its angular velocity ω_p determined over time intervals of 10 thousand years, fluctuates with an average period of 87 thousand years over a wide range from negative to positive values. Its average value is $\omega_{pm} = 1672$ ''/century. During the epochs of the retrograde perihelion rotation, the Mars eccentricity e is practically zero, and the orbit is a circle. Thus, all oscillation periods and their amplitudes in Fig. 2 were repeated without change over a longer time interval of 10 million years. In addition, new oscillations with large periods never appear, i.e., the oscillations of the Mars orbit occur in stationary mode.

4.3. Results for the last 100 million years

Figure 4 shows the evolution of Mars' orbital parameters over the period of the past 100 million years with first-type initial conditions. To avoid increasing the scale for the perihelion angle φ_p , this angle is represented here over the interval of the last 50 million years as $\varphi_p = \varphi_p + 4000$. A linear approximation with a period of $T_p = 78$ thousand years is plotted as a thin line on the graph of perihelion angle. At this scale, this approximation is indistinguishable from the φ_p graph. In the solutions, the interval between output points was 10 thousand years, i.e., twice as large as in the previous solution. This resulted in a slight difference in the average angular velocity ω_{pm} of perihelion motion. Overall, all periods and amplitudes of orbital-parameter oscillations remain unchanged. Some differences in the last digits are due to different purposes used at different times when recording results.

4.4. Comparison of our results with the results by other researchers

In the paper [12], we compared our results with the results obtained by other authors. Based on the secular perturbation theory [2], Cohen et al. (1973) [27] calculated the evolution of the Mars' orbit over the period of the last 5 million years. In a later paper [6], this evolution was traced over a longer time interval and with greater accuracy. The latter is confirmed by the fact



that the results of the above paper coincide with our results over a longer time interval, for example, for the eccentricity e , as far back in the past as

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6 million years ago, the inclination angle i as far back in the past as 30 million years ago, and the perihelion angle φ_p as far back in the past as 10 million years ago. However, over longer time intervals, these results begin to diverge from ours. For example, in the work by [28] it is noted that the Mars orbit eccentricity can exceed 0.2, and the chaotic diffusion of Mercury is so large that its eccentricity can potentially reach a value close to unity, and this planet can be ejected from the Solar system. Already at time intervals of 10 million years, weak discrepancies appear for the Earth orbit [29], which, according to these authors, are caused by multiple resonances occurring in the Solar system. Due to those resonances, the motion of the Solar system is chaotic [28, 30]. Therefore, the motion of the Earth [29] and Mars [6] cannot be calculated with acceptable accuracy for a time interval greater than 20 Myr.

In fact, as Figure 4 shows, over the interval of 100 million years the Mars orbits parameters fluctuate within constant limits. The oscillations shown in Figure 3 over a 3-million-year interval are repeated many times over the next 97 million years, indicating that the Mars orbit is stable with no tendency to changing. We have obtained similar results for other planets [8, 16] as well. Those results were confirmed using three sets of initial conditions, indicating that those studies have established the stability of both the orbits and the Solar system as a whole.

The current solution additionally took into account the minor planets closest to Mars, - Ceres, Pallas, and Vesta. The initial conditions were for the year 2000, and the velocities of the bodies were refined. Despite these refinements to the initial conditions, the results of the new solution for the Mars orbit proved to be almost perfectly coincident with those of the previous solution. The following conclusions follow from here. (i) The Solar system is a stable formation, and small changes in initial conditions do not alter its evolution. (ii). Other minor bodies in the Solar system exert no significant influence on the Mars orbits evolution; (iii) The results for the Solar-system evolution over the period of the last 100 million years, obtained taking into account only the Sun and the planets provide reliable results even without considering a large number of small bodies like asteroids and comets.

The manifestations of instability and chaos in planetary motion described in the works by other authors are most likely due to the solution methods those authors used. Furthermore, it should also be emphasized that the resonances and instabilities noted in the literature appear in the simplified equations of motion when solved using approximate analytical methods. Such phenomena never arise when integrating unsimplified equations (1) using the method we used [9].

4.5. Precession of the orbital axis

4.5.1. Orbital axis and its projections

When analyzing the results on the evolution of the orbits of all planets, it was established [17, 26] that for all planets the ascending-node oscillations occur near the same value of $\varphi_{\Omega m} = 0.068$, i.e. there is a direction in space common for all the planets. Since the problem is being solved for the Solar system isolated from external influences, this direction can only be defined by the position of the angular momentum $\vec{M}$ of the entire Solar system. The projections of that momentum onto the coordinate axes (see Fig. 1) in dimensionless units for the third-type initial conditions have the values:

$$\begin{aligned} M_x &= 1.08174716354836 \times 10^{-5}, \\ M_y &= -1.60632882219213 \times 10^{-4}, \\ M_z &= 3.79121141127551 \times 10^{-4}. \end{aligned} \quad (4)$$

The plane whose axis is the momentum vector $\vec{M}$ is called the Laplace plane [3]. The position of this plane relative to the $Oxyz$ coordinate system is defined by the Euler angles (see Fig. 1), which can be determined as follows:

$$i_M = \arccos(M_z/M) = 0.401581; \varphi_M = \pi/2 + \arctan(M_y/M_x) = 0.0672413, \quad (5)$$

where $M = \sqrt{M_x^2 + M_y^2 + M_z^2} = 4.11889281 \times 10^{-4}$. In dimensional form, the moment module $M = 3.13338111 \times 10^{43} \text{ kg m}^2/\text{sec}$.

In Fig. 1, the Laplace plane is shown with the arc segment $\gamma_M A_1$, and the definition of the angles i_M and φ_M is given in the caption. The angular-momentum vector $\vec{M}$ is determined by the composition of bodies in the Solar system. For the first-type initial conditions, the angles (5) were slightly different: $i_{M1} = 0.401833$; $\varphi_{M1} = 0.068094$. As evident, the angle of the ascending node φ_M of the Laplace plane, reckoned in the $Oxyz$ plane, coincides with the identified direction of $\varphi_{\Omega m}$, around which the ascending nodes of the planets oscillate.

We introduce a fixed coordinate system $Ox_M y_M z_M$ (Fig. 1) related with the Laplace plane. We direct the z_M -axis along the angular-momentum vector $\vec{M}$ and introduce a unit vector of the planet's orbital axis $\vec{S}_{pl}$ directed perpendicular to its plane [17, 26]. The planetary orbits are not flat and closed trajectories. To find the average orbital plane of a planet over one revolution, we used several algorithms, among which the following proved to be the most successful. We direct the unit vector of the orbital axis $\vec{S}_{pl}$ along the vector $\vec{M}_p$, the average angular momentum of the planet over one revolution. Then, we have:

$$S_{px} = M_{px}/M_p, \quad S_{py} = M_{py}/M_p, \quad S_{pz} = M_{pz}/M_p, \quad (6)$$

where M_{px} , M_{py} , and M_{pz} are the projections of the planet's angular momentum onto the axes of the equatorial heliocentric coordinate system;

$$M_p = \sqrt{M_{px}^2 + M_{py}^2 + M_{pz}^2};$$

$$S = \sqrt{S_{px}^2 + S_{py}^2 + S_{pz}^2} = 1.$$

If the angles i and φ_Ω defining the position of the orbital plane in the equatorial coordinate system xyz have already been found, then the projections of the orbital axis can be calculated using the following algorithm:

$$S_z = \cos i; \quad S_y = \sqrt{1 - S_z^2} \cos \varphi_\Omega; \quad S_x = -S_y \tan \varphi_\Omega. \quad (7)$$

The projections of the orbital axis $\vec{S}_{pl}$ onto the axes of the inertial system $x_M y_M z_M$ are determined by the relations [17, 26]:

$$\begin{aligned} S_{xM} &= S_{px} \cdot \cos \varphi_M + S_{py} \cdot \sin \varphi_M; \\ S_{yM} &= S_{px} \cdot \sin \varphi_M \cdot i_M + S_{py} \cdot \cos \varphi_M \cdot \cos i_M + S_{pz} \cdot \sin i_M; \\ S_{zM} &= S_{px} \cdot \sin \varphi_M \cdot \sin i_M + S_{py} \cdot \cos \varphi_M \cdot \sin i_M + S_{pz} \cdot \cos i_M. \end{aligned} \quad (8)$$

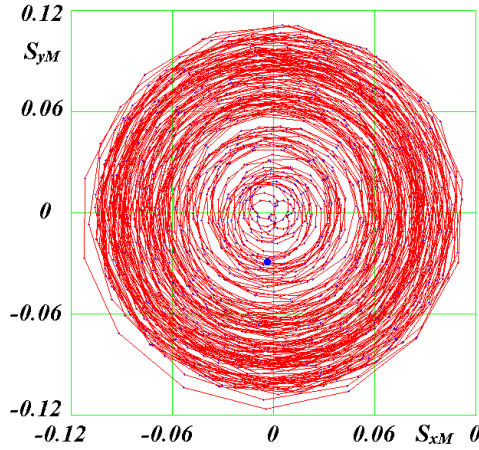


Figure 5. Precession of Mars' orbital axis $\vec{S}_{pl}$ during the last 10 million years projected onto the Laplace plane $y_M x_M$: In the graph, the interval between the dots is 5 thousand years; the large dot shows the position of Mars' orbital axis in the epoch of 2000.0.

4.5.2. Precession of the orbital axis

Figure 5 shows the evolution of the orbital axis $\vec{S}_{pl}$ in the $y_M x_M$ plane, perpendicular to the angular-momentum vector $\vec{M}$. The initial position of the

end of the orbital axis is shown with a large dot. From this dot on, the axis moves counterclockwise into the past and clockwise into the future. Over 10 million years, these movements are symmetrical about the origin, onto which the vector $\vec{M}$ is projected. Thus, the orbital axis $\vec{S}_{pl}$ rotates around the momentum vector clockwise, i.e., counterclockwise to the Mars' orbital motion around the Sun. The average period of precession of the axis over the period of 10 million years is $T_S = -73.0696$ kyr, and the average angular velocity is $\omega_{Sm} = -1774$ ''/century. The trajectory of the end of the vector $\vec{S}_{pl}$ in the $x_M y_M$ -plane is an open curve that spreads from the central to peripheral region in a few revolutions. This is due to variations of the angle θ_s (see Fig. 1) of the deviation of the orbital axis $\vec{S}_{pl}$ from the momentum vector $\vec{M}$, which is determined as follows:

$$\theta_S = \arccos S_{zM}. \quad (9)$$

The oscillations of the axis $\vec{S}_{pl}$ will be called nutational. There are two periods of nutational oscillations: $T_{\theta 1} = 150.6$ kyr and $T_{\theta 2} = 1.163$ Myr, whose amplitudes are respectively equal to: $\theta_{A1} = 9.654 \times 10^{-3}$ and $\theta_{A2} = 2.885 \times 10^{-2}$ radians. These periods are much greater than the period of precession of the T_S axis, therefore, long-term movements of the orbital axis from center to periphery, shown in Fig. 5, occur during many revolutions of the orbital axis around the momentum vector $\vec{M}$. In this case, the nutation angle θ_s varies in the range $1.10 \times 10^{-3} < \theta_s < 0.117$ radians with an average value $\theta_{s,m} = 0.0711$ radians. The maximum deviation of the axis $\vec{S}_{pl}$ from the momentum $\vec{M}$ is $\theta_{s,max} = 6.682^\circ$, and the range of nutational oscillations reaches $2 \cdot 6.682^\circ = 13.36^\circ$.

Thus, the complex oscillations of the ascending-node angle φ_Ω of the orbit and its inclination angle i , shown in Fig. 3, are caused by the precessional motion of the orbital axis $\vec{S}_{pl}$ around the momentum vector $\vec{M}$ with a period $T_S = -73.0696$ kyr and its nutational oscillations with periods $T_{\theta 1}$ and $T_{\theta 2}$.

It should be noted that, in the graph of Fig. 5, the trajectory of the axis $\vec{S}_{pl}$ is depicted with straight lines connecting the dots. Since the interval between dots is 5 kyr, then with a precession period of $T_S = -73.0696$ kyr the trajectory is represented by broken lines.

4.5.3. Evolution of precession angles

As it was noted above, the deviation of the orbital-axis vector $\vec{S}_{pl}$ from the angular-momentum vector $\vec{M}$ is determined by the nutation angle θ_s according to formula (9), and the precession of this vector relative to $\vec{M}$ is given by the precession angle [8]

$$\psi_S = \arctan S_{yM}/S_{xM} + 0.5\pi, \quad (10)$$

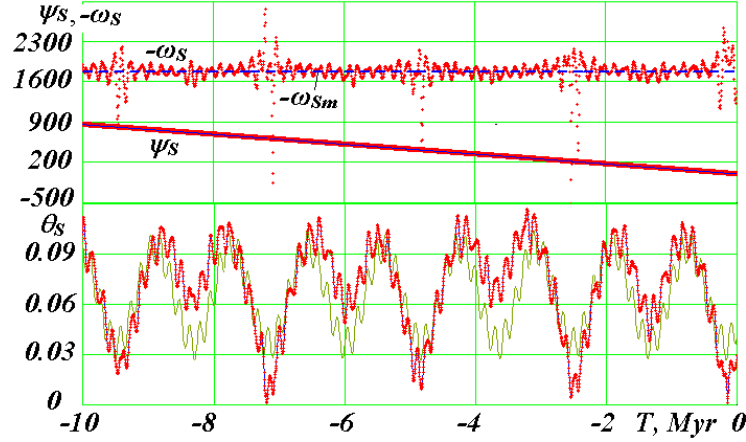


Figure 6. Evolution of the precession angle ψ_S and the angle θ_S of inclination of the Mars orbital plane relative to the momentum $\vec{M}$ over time intervals of 10 Myr into the past: ω_S is the angular velocity of rotation of the orbital axis $\vec{S}_{pl}$ in $''/\text{century}$ units over time intervals of 10 thousand years; the average angular velocity of the orbital-axis rotation is $\omega_{Sm} = -1774''/\text{century}$; the thin lines 1 and 2 show the approximation dependences ψ_{Sap} and θ_{Sap} .

which is reckoned (see Fig. 1) in the $x_M O y_M$ plane from the x_M -axis to the ascending node of Mars' orbit in the $x_M O y_M$ plane. Since the period of the arctan function is equal to π , when in the case of $\psi_S > \pi$ the signs of S_{yM} and S_{xM} must be taken into account. Taking these signs into account, the calculation using formula (10) yields an angle ψ_S varying from 0 to 2π , i.e., as a discontinuous value. The differences in the angle values at adjacent points are then determined, and by summing the differences, the angle ψ_S is converted into a continuous value. The variation of the precession angle ψ_S and that of the angle of inclination θ_S of Mars' orbit are shown in Fig. 6 over 10 million years. Changes in the angle ψ_S are not monotonic, and along with its decrease into the future (the axis S_{pl} rotates clockwise) there are also sections with increasing angle ψ_S . However, as shown by the dots in Fig. 6, over a large time interval of 10 million years these precession irregularities are not visible. Here, the thin line 1 plots the approximate linear relation:

$$\psi_{Sap} = \psi_{S0} + 2\pi \cdot T/T_S, \quad (11)$$

where $\psi_{S0} = 3.022$ is the precession angle at the initial time, i.e., at $T = 0$; and T_S is the precession period of the orbital axis.

As seen from the graphs, the values of ψ_{Sap} and ψ_S are coincident. The unevenness of the precession of the axis $\vec{S}_{pl}$ is evident from its angular velocity ω_S , which is determined over 10 thousand year time intervals. There are small

fluctuations of ω_S with a period of 150 thousand years, as well as individual spikes that occur with a periodicity of 2.3 million years. As already noted, the average precession velocity is $\omega_{Sm} = -1774''/\text{day}$.

The variation of the inclination angle θ_s is shown in Fig. 6 with dots. As already noted, two periods of its oscillations can be distinguished. The thin line 2 shows the approximation dependence

$$\theta_{Sap} = \theta_{S0} + \theta_{SA1} \cdot \sin(\varphi_{\theta 1} + 2\pi \cdot T/T_{\theta 1}) + \theta_{SA2} \cdot \sin(\varphi_{\theta 2} + 2\pi \cdot T/T_{\theta 2}), \quad (12)$$

where $\theta_{S0} = 6.53 \times 10^{-2}$ radians; $\varphi_{\theta 1} = -0.942$; and $\varphi_{\theta 2} = -0.628$ radians.

The approximation of θ_{Sap} reflects the main oscillations of the inclination angle θ_s , but does not completely coincide with it. The precession angle ψ_S exhibits similar oscillations, but their amplitude is significantly larger.

Similar precessional rotations and nutational oscillations occur in the orbital axes of other planets in the Solar system [8, 16].

Given the angles ψ_S and θ_s , one can determine the angles i and φ_Ω of the orbital plane with respect to the equatorial coordinate system xyz (Fig. 1). In the spherical triangle $\gamma_M \gamma_2 A_1$, two angles $\gamma_M = i_M$ and $A_1 = \theta_S$, and the adjacent side $\gamma_M A_1$ are known, as indicated in the caption to Fig. 1. Using the cosine theorem for angles (p. 31, [3]), one can find the angle $\gamma_2 = \pi - i$, from which we obtain an expression for the angle

$$i' = \pi - \arccos(-\cos i_M \cdot \cos \theta_S + \sin i_M \cdot \sin \theta_S \cdot \cos \psi_S). \quad (13)$$

For the side $\gamma_M \gamma_2$ of the triangle $\gamma_M \gamma_2 A_1$ we have: $\gamma_M \gamma_2 = \varphi_\Omega - \varphi_M$, then, according to the law of sines for spherical triangles, we obtain an expression for the ascending-node angle:

$$\varphi'_\Omega = \varphi_M + \arcsin[\sin \psi_S \cdot \sin \theta_S / \sin(\pi - i')]. \quad (14)$$

In Fig. 2, the dots of the angles φ_Ω and i are connected with a thin line of angles φ'_Ω and i' . Evidently, those angles are perfectly coincident. Thus, the precessional ψ_S and oscillatory θ_S motions of the orbital axis $\vec{S}_{pl}$ completely determine the oscillatory motions of the orbital plane relative to the Earth equators plane.

Substitution of approximating dependences (11) and (12) for ψ_{Sap} and θ_{Sap} , respectively, into equations (13) and (14) instead of the angles ψ_S and θ_S yields approximations of the angles i_{ap} and $\varphi_{\Omega ap}$. Those approximations do not exactly coincide with the fluctuations of the angles φ_Ω and i in Fig. 2, but they reflect the nature of the fluctuations. The advantage of the approximations of ψ_{Sap} and θ_{Sap} is that they can be used without time limitation.

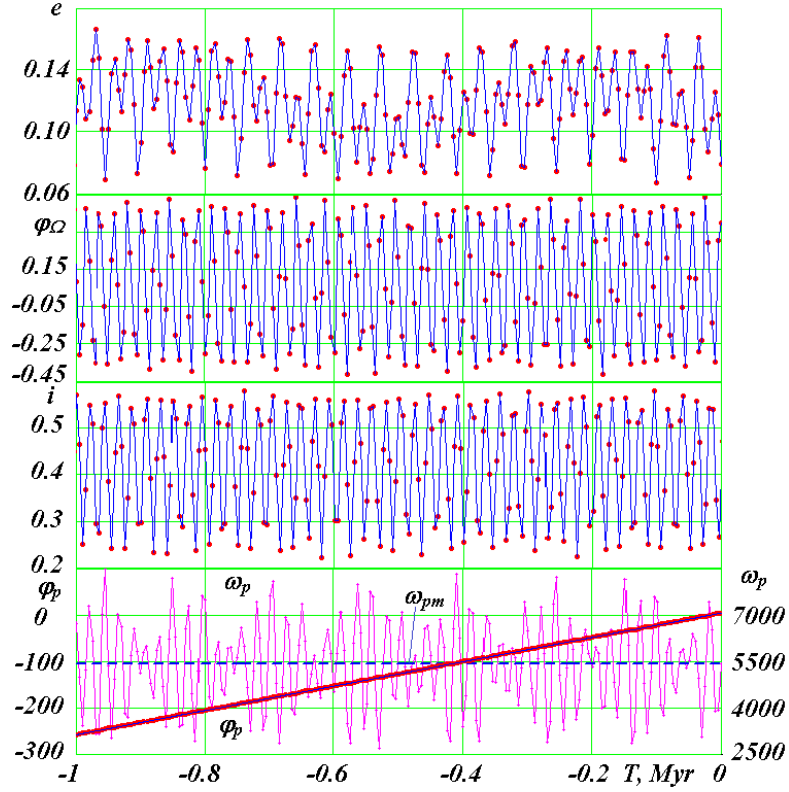


Figure 7. Evolution of Ceres' orbit over the last 1 million years; the average angular velocity of perihelion rotation is $\omega_{pm} = 5420$ ''/century; for the rest designations see Fig. 2.

5. Evolution of the orbits of minor planets

5.1. Evolution of the orbits of minor planets over the last 1 million years

Since the periods of oscillations of the orbital parameters of minor planets are considerably shorter than the oscillation periods of Mars orbit, we will consider the evolution of the orbits of these planets over a million-year time interval. The evolution of Ceres's orbit over a time of the last 1 million years is shown in Fig. 7. The eccentricity e undergoes short-period oscillations with a first period $T_{e1} = 26$ thousand years and an amplitude of 0.043. There is also a second period of eccentricity oscillations $T_{e2} = 640$ thousand years with an amplitude of 0.0097. However, these oscillations are not harmonic. The

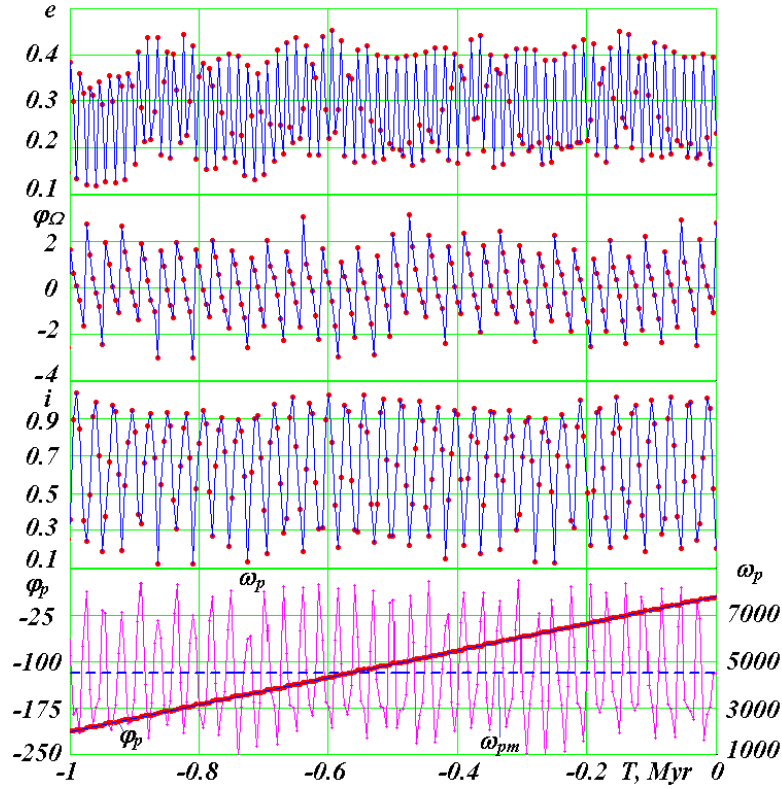


Figure 8. Evolution of Pallas' orbit over the last 1 million years: the average angular velocity of perihelion rotation is $\omega_{pm} = 4510$ ''/century; for the rest designations see Fig. 2.

average value of eccentricity is $e_m = 0.118$.

The longitude of the ascending node φ_Ω varies with an average period of $T_\Omega = -T_S = 21.9$ thousand years around the average value $\varphi_{\Omega m} = 0.067$ radians. The angle i of inclination of the orbital plane to the equator plane of 2000.0 experiences oscillations with the same period $T_i = -T_S = 21.9$ thousand years around the average value $i_m = 0.419$ radians. The angle φ_p that defines the perihelion position increases almost linearly with time into the future over the interval of 1 million years, i.e. the perihelion rotates in the direction of Ceres's revolution around the Sun, completing on the average one revolution in every $T_p = 23.9$ thousand years. The graph shows small variations of the perihelion angle. A linear approximation with a period T_p is shown in the graph by a thin line. It does not deviate from the φ_p graph. The angular velocity of perihelion ω_p oscillates with the main period of 20.8

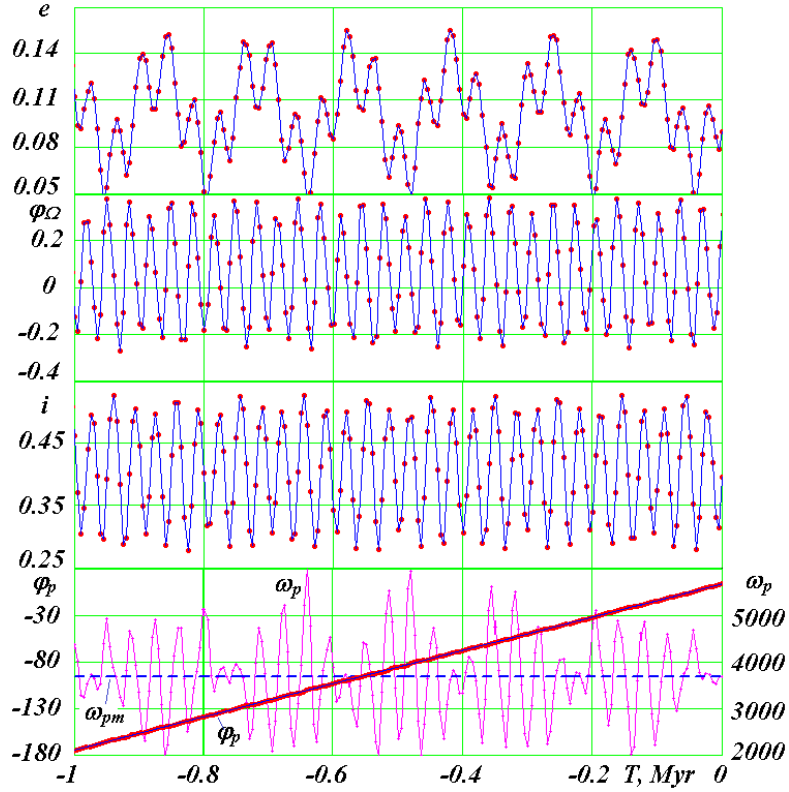


Figure 9. Evolution of Vesta's orbit the last 1 million years: $\omega_{pm} = 3687$ "/century is the average angular velocity of perihelion rotation; for the remaining notation see Fig. 2.

thousand years with an average value of $\omega_{pm} = 5424$ "/century.

Compared to Mars, the oscillations of Ceres' orbital parameters are more regular, with the exception of eccentricity e , and the oscillation periods are several times shorter.

The evolution of Pallas' orbit over the last 1 million years is shown in Fig. 8. The eccentricity e undergoes short-period oscillations with a main period $T_{e1} = 14.3$ thousand years and an average amplitude of 0.14. There is also a second period of eccentricity oscillations $T_{e2} = 231$ thousand years with an amplitude of 0.02. These oscillations are also non-harmonic. The average value of eccentricity is $e_m = 0.287$.

Unlike the orbital plane of the preceding bodies, the plane of the Pallas orbit is inclined to the Earth's equator (Fig. 1) at a large angle i , so the longitude φ_Ω of the ascending node varies rather widely. The average period

of oscillations of that node is $T_\Omega = T_S = 27.9$ thousand years. With this period, the longitude φ_Ω fluctuates around the average value $\varphi_{\Omega m} = 0.0403$ radians. This value of $\varphi_{\Omega m}$ differs from the value $\varphi_{\Omega m} = 0.067$, characteristic of all planets, by a small amount. This error is due to the large range of φ_Ω oscillations and the small number of output points. As already noted, these points follow each other every 5 thousand years.

The angle i of inclination of the orbital plane to the equatorial plane of 2000.0 yr experiences oscillations with the same period $T_i = -T_S = -27.9$ thousand years around the average value $i_m = 0.644$ radians. Over the interval of 1 million years, the angle φ_p that defines the perihelion position increases almost linearly with time varied into the future, i.e. the perihelion moves in the direction of Pallas's revolution around the Sun, completing on the average one revolution in $T_p = 28.7$ thousand years. The angular velocity of perihelion ω_p oscillates with a main period of 28 thousand years with an average value $\omega_{pm} = 4510$ ''/century. The perihelion oscillations of Pallas are also more regular compared to Mars.

The evolution of Vesta's orbit over the last 1 million years is shown in Fig. 9. The eccentricity e undergoes short-period oscillations with a first period $T_{e1} = 39.9$ thousand years and an amplitude of 0.036. There is also a second period of eccentricity oscillations $T_{e2} = 150$ thousand years with an amplitude of 0.030. These oscillations are harmonic. The average value of eccentricity is $e_m = 0.104$.

The longitude of the ascending node φ_Ω varies with an average period of $T_\Omega = -T_S = 32.7$ thousand years around an average value of $\varphi_{\Omega m} = 0.067$ radians. The angle i of inclination of the orbital plane to the equator plane of 2000.0 oscillates with the same period $T_\Omega = -T_S = 32.7$ thousand years around an average value of $i_m = 0.409$ radians. Over an interval of 1 million years, the angle φ_p that defines the perihelion position increases almost linearly with the change of time into the future, i.e. the perihelion moves in the direction of Vesta's revolution around the Sun, completing, on the average, one revolution in $T_p = 35.1$ thousand years. The angular velocity of perihelion ω_p oscillates with a main period of 32 thousand years around the average value $\omega_{pm} = 3687$ ''/century.

5.2. Evolution of the orbits of minor planets the last 10 million years

The periods, the amplitudes, and the average values of the orbital parameters of minor planets given above for 1 million years were determined over a 10-million-year interval. Therefore, they remain the same in the following graphs. Since the oscillation periods are small compared to this interval, the oscillations merge into a continuous opaque band. Therefore, when plotting these results, various techniques must be used to ensure distinguishable results.

Figure 10 shows the evolution of Ceres' orbital parameters over the period

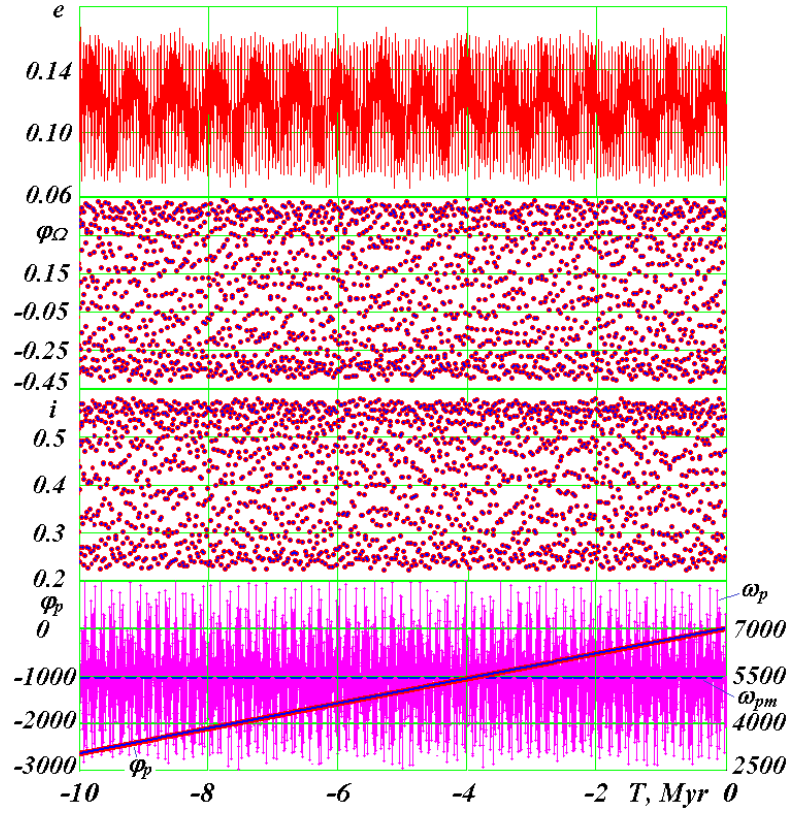


Figure 10. Evolution of Ceres' orbit over the last 10 million years. For designations, see Fig. 7.

of 10 million years. The second period of its oscillations, $T_{e2} = 640$ thousand years, is visible considering the condensation of eccentricity changes. The ascending-node angle φ_{Ω} and the inclination angle i are represented by dots. Their boundaries indicate that the oscillation amplitude remains unchanged.

The perihelion angle φ_p varies with the same period $T_p = 23.9$ thousand years, and the linear approximation does not deviate from the change of the angle φ_p . Thus, Ceres' orbital parameters fluctuate within constant limits. The previously presented periods and amplitudes over a 1-million-year interval remain unchanged.

Figure 11 shows the evolution of Pallas's orbital parameters the last 10 million years. Like in Fig. 10, the parameter scales on the axes, with the exception of the perihelion angle φ_p , are the same as those over the period of 1 million years. This indicates that the oscillations of those parameters remain

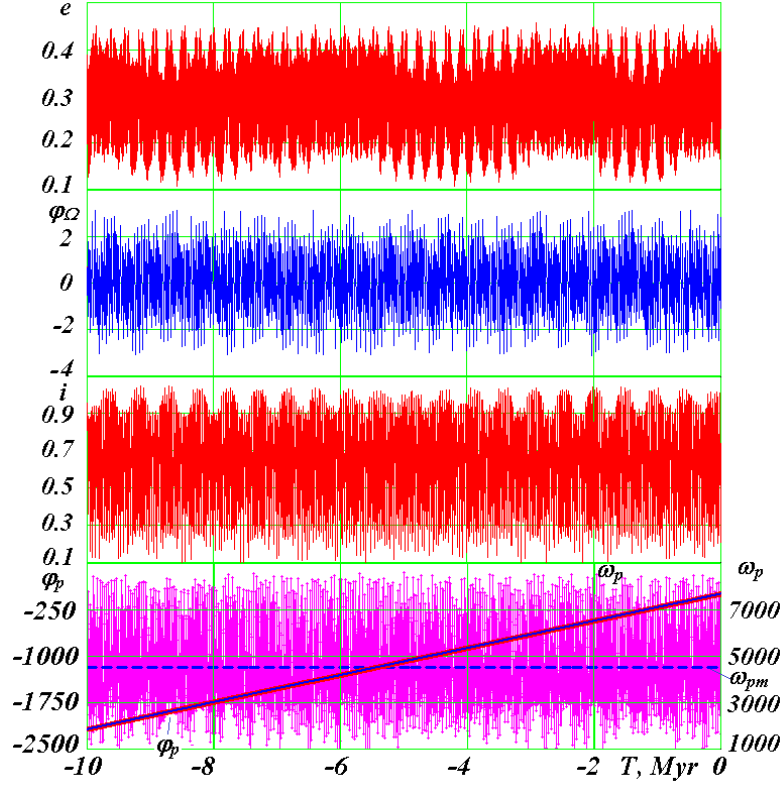


Figure 11. Evolution of Pallas's orbit over the last 10 million years. For designations, see Fig. 6.

constant. The second period of their oscillations, $T_{e2} = 231$ thousand years, is evident from the condensation of eccentricity data.

The perihelion angle φ_p varies with the same period $T_p = 28.7$ thousand years, and the linear approximation for it does not deviate from the change of the angle φ_p . Thus, the Pallas orbital parameters fluctuate within constant limits. The established periods and amplitudes of those oscillations over the 1 million year interval remain unchanged.

Figure 12 shows the evolution of Vesta orbital parameters over the last 10 million years. The second period of its oscillations, $T_{e2} = 150$ thousand years, is visible from the upper and lower boundaries of eccentricity e . The angles φ_Ω and i , and the angular velocity ω_p vary within the same limits as over the interval of 1 million years.

The perihelion angle φ_p varies with the same period $T_p = 35.1$ thousand years, and the linear approximation does not deviate from the change in the

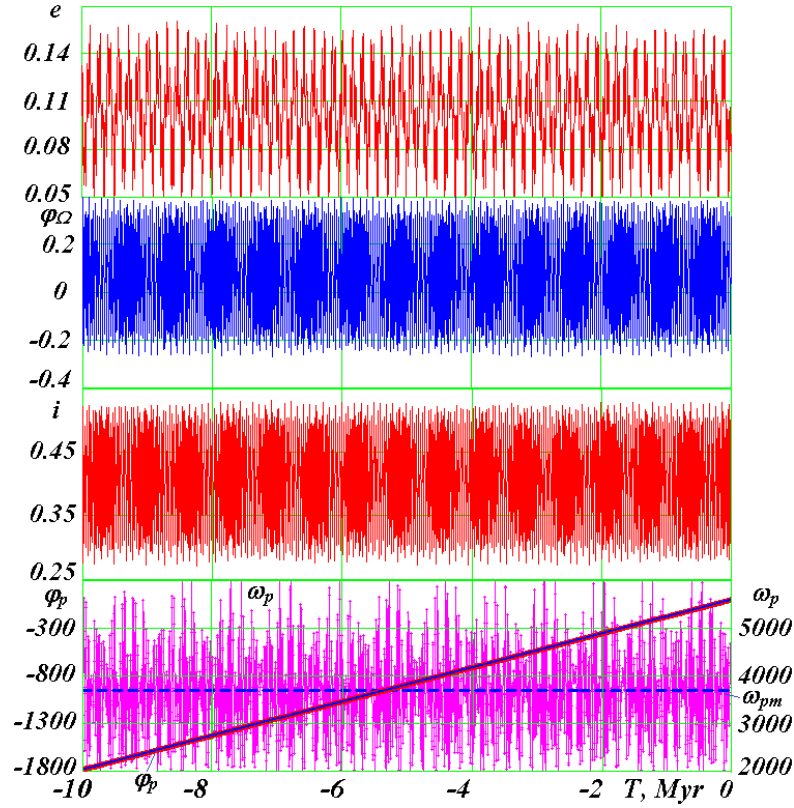


Figure 12. Evolution of Vesta's orbit over the last 10 million years. For designations, see Fig. 7.

angle φ_p . Thus, Vesta's orbital parameters fluctuate within constant limits. The established periods and amplitudes over the interval of the last 1 million years remain unchanged.

5.3. Precession of the orbital axes of minor planets

Figure 13a shows the precession of Ceres' orbital axis $\vec{S}_{pl}$ projected onto the $y_M x_M$ -plane, perpendicular to the angular momentum vector $\vec{M}$ (Fig. 1) over 2 million years. The projections of the end of the axis $\vec{S}_{pl}$ are shown with dots. The interval between the dots, as already noted, is 5 thousand years, which value is comparable with the period of precession of the axis $T_S = 21.9$ thousand years. Therefore, the segments connecting the dots form a broken line that covers an area larger than the area occupied by the end of the axis

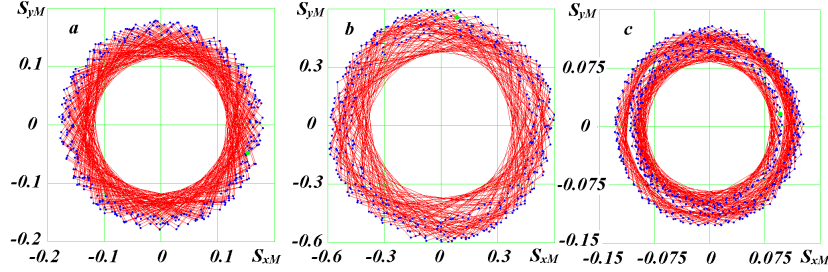


Figure 13. Precession of the orbital axes of Ceres (a), Pallas (b), and Vesta (c) over periods of several million years; the large dots show the positions of the orbital axes in the epoch 2000.0. For designation, see Fig. 5.

$\vec{S}_{pl}$. Thus, the end of the axis moves in the annular region occupied by the dots. The image of the precession in Fig. 13a is limited to 2 million years, since after 10 million years the movements become indistinguishable.

The position of the end point of the orbital axis $\vec{S}_{pl}$ at the initial time is shown by a large dot. From this dot on, the axis moves counterclockwise into the past and clockwise into the future. Over the time interval of the last 10 million years, these movements are symmetrical about the origin, onto which the vector $\vec{M}$ is projected. Thus, the orbital axis $\vec{S}_{pl}$ rotates around the vector of the angular momentum vector $\vec{M}$ in clockwise direction, i.e., in the direction opposite to the orbital motion of Ceres around the Sun. As already noted, the average precession period of the axis $\vec{S}_{pl}$ over 10 million years is $T_S = -21.872$ thousand years, and the average angular velocity is $\omega_{Sm} = -5925$ "/century.

The inclination angle θ_s varies within the range $0.1533 < \theta_s < 0.1835$ radians with an average value $\theta_{sm} = 0.1685$ radians. The maximum deviation of the axis from the momentum vector $\vec{M}$ makes an angle $\theta_{smax} = 10.51^\circ$, and the range of the nutational oscillations reaches $2 \cdot 10.51^\circ = 21^\circ$. The main period of those oscillations is $T_{\theta 1} = 39.4$ kyr with an amplitude of $\theta_{A1} = 1.342 \times 10^{-2}$ radians.

The period $T_{\theta 1}$ is almost twice as large as the period T_S of axis precession; that is why the movements of the orbital axis in Fig. 13a from the inner edge of the ring of the dots of the axis $\vec{S}_{pl}$ to the outer edge occurs during two revolutions of the axis.

The oscillations of the ascending node φ_Ω of the orbit and its inclination angle i shown in Fig. 7 are due to the precessional motion of the Ceres orbital axis $\vec{S}_{pl}$ around the momentum vector $\vec{M}$ with a period T_S and its nutational oscillations with periods $T_{\theta 1}$.

Figure 13b shows the precession of the Pallas orbital axis $\vec{S}_{pl}$ around the

vector $\vec{M}$, also over the time interval of 2 million years. The end of the axis $\vec{S}_{pl}$ moves in the ring region occupied by the dots, with a period of $T_S = -27.884$ kyr. The average angular velocity is $\omega_{Sm} = -4648$ "/century.

The inclination angle θ_s varies within the range $0.4850 < \theta_s < 0.6545$ radians with an average value of $\theta_{sm} = 0.5788$ radians. The maximum deviation of the axis $\vec{S}_{pl}$ from the momentum $\vec{M}$ makes an angle $\theta_{smax} = 37.5^\circ$, and the range of nutational oscillations reaches $2 \cdot 37.5^\circ = 75^\circ$. The main period of nutational oscillations is $T_{\theta 1} = 15.05$ kyr with an amplitude $\theta_{A1} = 5.085 \times 10^{-2}$ radians.

The period $T_{\theta 1}$ is almost two times smaller than the period T_S of precession of the axis, therefore, during one revolution of the orbital axis $\vec{S}_{pl}$ it passes almost twice from the inner edge of the ring of dots of the axis to the outer edge.

Figure 13c shows the precession of the Vesta orbital axis around the vector of the angular momentum over the period of 3 million years. The end of the axis moves in the ring region occupied by the dots with the period $T_S = -32.707$ kyr. The average angular velocity is $\omega_{Sm} = -4382$ "/century.

The inclination angle θ_s varies within the range $0.0919 < \theta_s < 0.1311$ radians with an average value of $\theta_{sm} = 0.1122$ radians. The maximum deviation of the axis $\vec{S}_{pl}$ from the momentum vector $\vec{M}$ makes an angle $\theta_{smax} = 7.51^\circ$, and the range of the nutational oscillations reaches $2 \cdot 7.51^\circ = 15^\circ$. The main period of nutational oscillations is $T_{\theta 1} = 97.69$ kyr with an amplitude of $\theta_{A1} = 1.868 \times 10^{-2}$ radians.

The period $T_{\theta 1}$ is three times as large as the precession period of the axis T_S , so the orbital axis $\vec{S}_{pl}$ traverses the distance from the inner to outer edge of the ring of dots in three revolutions. In Fig. 13c, the ring of dots is mainly stratified into two rings. This stratification is caused by the frequency of the points following each 5 thousand years and the frequency of the axis passing through the width of the ring in one revolution.

The oscillations of the ascending node φ_Ω of the orbit and its angle of inclination i shown in Figs. 8 and 9 are caused by the precessional motion of the axes $\vec{S}_{pl}$ of the Pallas and Vesta orbits, respectively, around the angular-momentum vector with periods T_S and their nutational oscillations with periods $T_{\theta 1}$.

5.4. The invariable plane of the Solar system

The orbital planes of the major and minor planets of the Solar system move relative to a fixed angular momentum $\vec{M}$ in space, and the Laplace plane perpendicular to it is the invariable plane of the Solar system. Therefore, for three centuries, beginning with P. S. Laplace, the celestial mechanics have proposed using it as a reference frame for coordinating astronomical objects.

However, the question of how accurately it can be determined in space had been raised. The answer to this question has now been established.

We determined the position of the invariable plane based on 14 main bodies of the Solar system and demonstrated its significance for the evolution of planetary motion. In the article [31], studies were conducted on the influence of these same 14 planets, as well as 718041 minor planets: asteroids, comets, and centaurs, on the position of the Laplace plane. Table 1 for the epoch of the year 2000.0, according to our data [33], shows the angles of its position i_M and $\varphi_{\Omega M}$ in radians relative to the plane of the equator and the angles of its position i_{Me} and Ω_M in degrees relative to the plane of the ecliptic. The angle of inclination i_{Me} and the longitude of the ascending node Ω_M relative to the plane of the ecliptic E_0E_0 (Fig. 1) are determined by formulas similar to (5), in which the projections of $\vec{M}$ onto the axes of the ecliptic coordinate system are calculated as follows:

$$M_{xe} = M_x; \quad M_{ye} = M_y \cos \epsilon_0 + M_z \sin \epsilon_0; \quad M_{ze} = -M_y \sin \epsilon_0 + M_z \cos \epsilon_0.$$

where the angle between the E_0E_0 and A_0A_0 planes at the epoch 2000.0 is $\epsilon_0 = 0.409092629689404$ [19].

Table 1. Laplace plane angles according to our research (Smulsky) and other authors research (Li et al., 2019 [31]).

Number of bodies, n	Smulsky				(Li et al., 2019)	
	i_M	$\varphi_{\Omega M}$	i_{Me}	Ω_M	i_{Me}	Ω_M
14	0.4015 80676	0.0672 41297	1°34'43" .33872	107°34'57" .02415	1°34'43" .34059	107°34'56" .21287
14 + 718041	-	-	-	-	1°34'44" .37736	107°33'26" .97123

It should be recalled that our results were obtained based on the DE422 ephemerides, in which the velocities of the bodies were modified, and the data were recalculated to the epoch of 2000.0. For comparison, Table 1 provides the angles of the Laplace plane i_{Me} and Ω_M relative to the ecliptic plane for the same 14 bodies according to the paper [31]. They were obtained from the DE405 ephemerides for November 8, 2016. Despite the different origins of these data, the angles i_{Me} and Ω_M of the Laplace plane, as can be seen from Table 1, practically coincide. The paper [31] provides the value of the angular momentum modulus for these bodies $M = 3.1333 \times 10^{43}$ kg m²/s with 5 significant digits. It coincides with our value of the moment $M = 3.13338111 \times 10^{43}$ kg m²/s, which we give with a large number of significant digits.

Let's consider how our results in Table 1 for 14 bodies differ from those of [31]. The inclination angle i_{Me} of the Laplace plane to the ecliptic plane

differs by 1.9 mas (milliarcseconds), and the longitude of the ascending node Ω_M differs by 811 mas. Thus, based on these studies, it was established that, with this error, the current dynamic characteristics of 14 bodies allow us to determine the invariant plane of the Solar system.

A second question appears: to what extent will taking into account the dynamic characteristics of other Solar system bodies in the future affect the position of the Laplace plane? This question is answered by the paper [31], which assessed the influence of 718041 minor planets on its position. As can be seen from Table 1, further consideration of other bodies in the Solar system may lead to a change in the inclination angle i_{Me} of the Laplace plane by 1".

Currently, in astronomical research, the reference frame is changed every 50 years. Due to the evolution of the planes of the Earth's orbit and its equator, the ascending node of the equator moves along the plane of the orbit at a rate of 50" per year. This means that over 50 years, the position of the new reference frame changes by 40, or almost 1°. In addition, the inclination angle ϵ of the equatorial plane to the plane of the Earth's orbit decreases at a rate of -0.468" per year. The recalculation of the coordinates of astronomical objects to a new reference system is carried out using an approximation of observation data for approximately 1000 years. These approximations reflect the linear trends in parameter change that have occurred over the past 10 thousand years. However, the epoch with a linear trend in change is coming to an end. The change in the angle ϵ is currently slowing, and in 1.88 thousand years, its rate of change will become zero, after which the obliquity ϵ will begin to increase [13]. Therefore, the formulas for recalculating coordinates to the new reference frame will need to be modified. Thus, with 50-year changes in the reference frame, the coordinates of astronomical objects change by approximately one degree and must be recalculated. Given the circumstances noted, recalculation will lead to errors that will be significantly greater than 1" over a series of 50-year periods.

Therefore, it makes sense to transition to a reference frame with an invariable Laplace plane in 2050 and set its validity period to 500 years. Then, the coordinates of astronomical objects will be considered in the invariable reference frame. This will improve the accuracy of observations, and over 500 years it will be possible to determine previously undetected movements and changes in distant astronomical objects.

After 500 years, the dynamic characteristics of many minor planets in the Solar System will be refined, and it will be possible to transition to new parameters for the Laplace plane. These will differ from the previous plane by several tens of mas. Therefore, recalculation to the new coordinates will not lead to the same errors as recalculation over 50-year periods.

It seems to us that the detailed development of these studies and the transition to an unchangeable frame of reference are the most important tasks of astronomical science in the next 25 years.

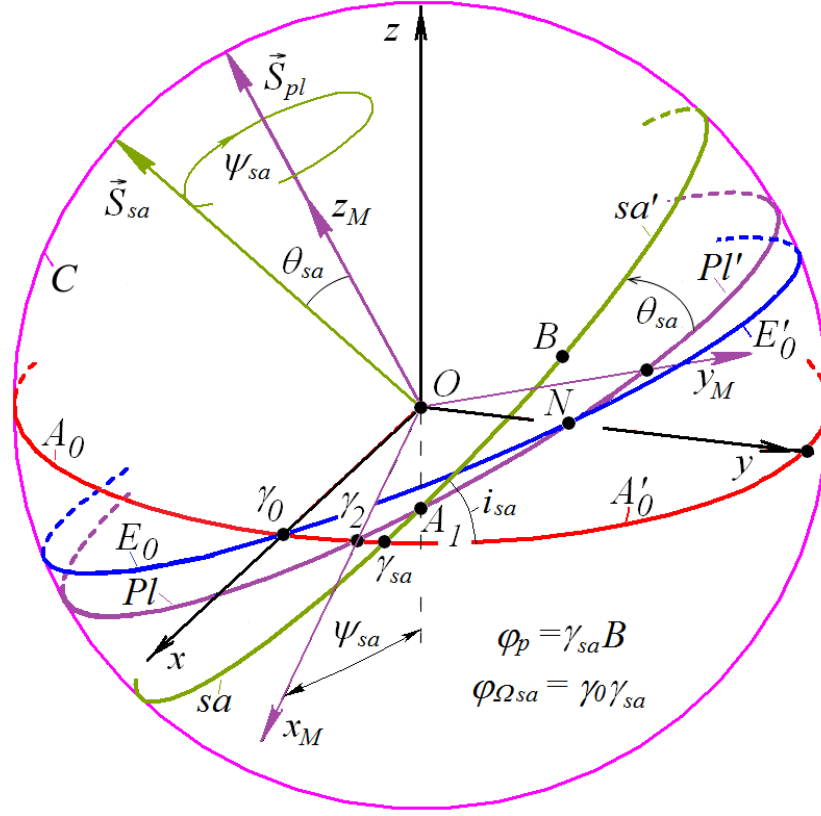


Figure 14. Parameters of the satellite orbit (sa) in the fixed equatorial system xyz : C is the celestial sphere; fixed planes at the epoch of 2000.0 with $JD_s = 2451545$ are as follows: $A_0 A'_0$ is the fixed plane of the Earth equator, $E_0 E'_0$ is the fixed plane of the Earth's orbit (plane of ecliptic); γ_0 is the vernal equinox point at the epoch of 2000.0; $PlPl'$ is the moving plane of the planet at the epoch T ; $sasa'$ is the moving plane of the satellites orbit at the epoch T ; $\vec{S}_{pl}$ and $\vec{S}_{sa}$ are the axes of the moving orbits of the planet $PlPl'$ and the satellite $sasa'$, respectively; B is the projection of the pericenter point of the satellites orbit on the celestial sphere, and $\varphi_p = \gamma_{sa} B$ is its angular position; angles defining the position of the planes of the satellite's orbit with respect to the Earth equatorial plane $A_0 A'_0$: i_{sa} is the angle of inclination and $\varphi_{\Omega sa} = \gamma_0 \gamma_{sa}$ is the angle of the ascending node of the orbit; ψ_{sa} and θ_{sa} are the angles of precession and inclination of the satellite's orbit relative to the moving plane of the planet's orbit.

6. Evolution of the orbits of Mars' satellites

6.1. Some details of the study

The orbits of Mars' satellites are also analyzed in the equatorial coordinate system (Fig. 14), but relative to Mars. Thus, Mars is at the origin O , and the plane A_0A_0 is the plane of Earth's equator at the epoch of 2000.0. The satellite's orbital plane $sasa$ is also defined by the ascending-node angle $\varphi_{\Omega sa}$ and the inclination angle i_{sa} . As we established for the Moon [32], the satellite's orbital axis $\vec{S}_{sa}$ precesses around the orbital axis $\vec{S}_{pl}$ of its planet, in this case, Mars. Therefore, the axes of the $x_M y_M z_M$ coordinate system, in which precession occurs, are associated with the planet's orbital axis $\vec{S}_{pl}$ and its plane. $PlPl'$.

A mathematical model of the Moon's motion and evolution was developed as a result of studying the Moon orbit evolution [32]. This model used the following parameters of the Moon orbit: the pericenter radius R_p , the eccentricity e , the orbital period P , and the angles φ_Ω , i and φ_p . The same parameters will also be used for the satellites of Mars.

The orbits of satellites change substantially with each revolution. Therefore, we studied the Moon's orbit over its 736 continuous revolutions, which occurred over a period of 56.7 years. We also studied the orbits of Mars' satellites over 700 continuous revolutions. We then examined their orbital evolution over a 6 thousand years interval, and then over an interval of 80 thousand years. The semi-major axes of the orbits of Mars's satellites a are much smaller than the semi-major axis of the Moon's orbit. Therefore, differential equations (1) were integrated using the Galactica program with a step size of $dT = 1 \times 10^{-6}$ year, which is 100 times smaller than the step size that was used in studying the evolution of the planets.

6.2. Dynamics and evolution of Phobos' orbit

The variation of Phobos's orbital parameters over its 700 continuous revolutions is shown in Fig. 15. As seen from the range of time T , these revolutions last 0.7 year. The pericenter radius R_p is shown by dots that merge into a solid thick line. The pericenter performs almost one oscillation with a period $T_{Rp1} = 0.9$ years and an amplitude $R_{pA1} = 0.13$ km around the mean value $R_{pm} = 9232.6$ km. The thin line shows an approximate sinusoid with this period. The eccentricity e also varies with this period of $T_{e1} = 0.9$ year and amplitude $e_{A1} = 1.4 \times 10^{-5}$. Those oscillations occur around its average value $e_m = 0.015094$. The thin line shows a sinusoid with this period.

The orbital period P is shown on this graph by small dots, and the thin

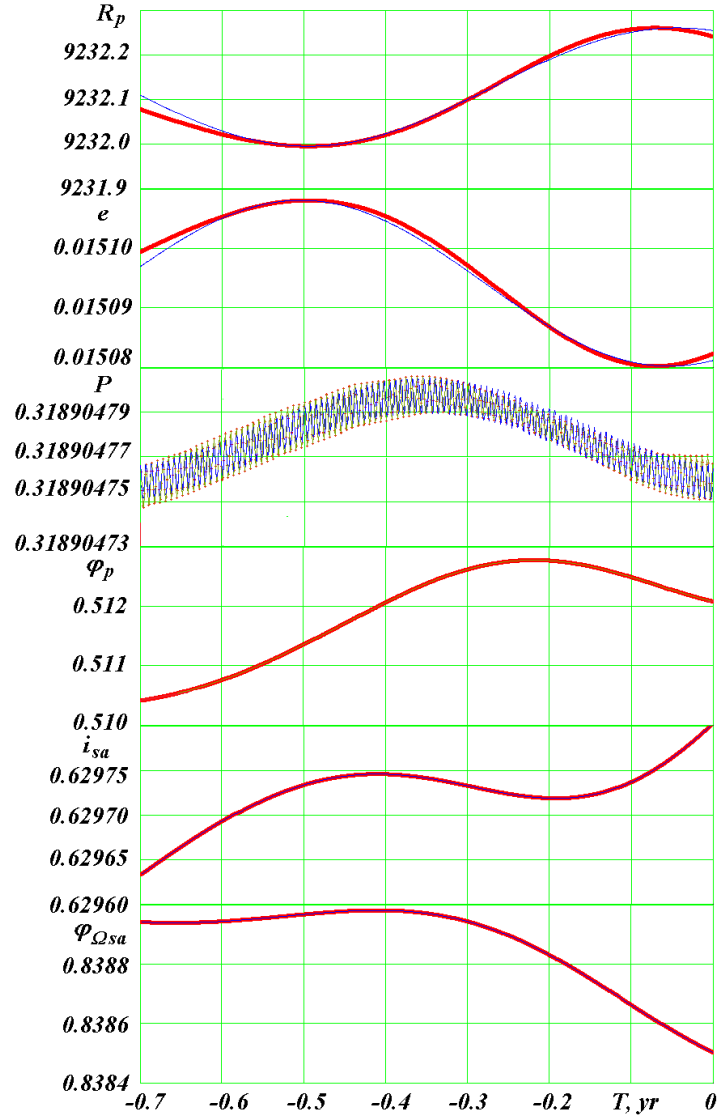


Figure 15. Dynamics of the orbital elements of Phobos for the time of 700 continuous revolutions in the coordinate system of the Earth's equator with the origin at the center of Mars: the pericenter radius R_p is given in kms, the period P in days and the time T in sidereal years with 365.25636042 days in each year; the angles φ_p , i_{sa} and φ_{sa} are in radians; the thin lines in the three upper graphs are approximate sinusoids.

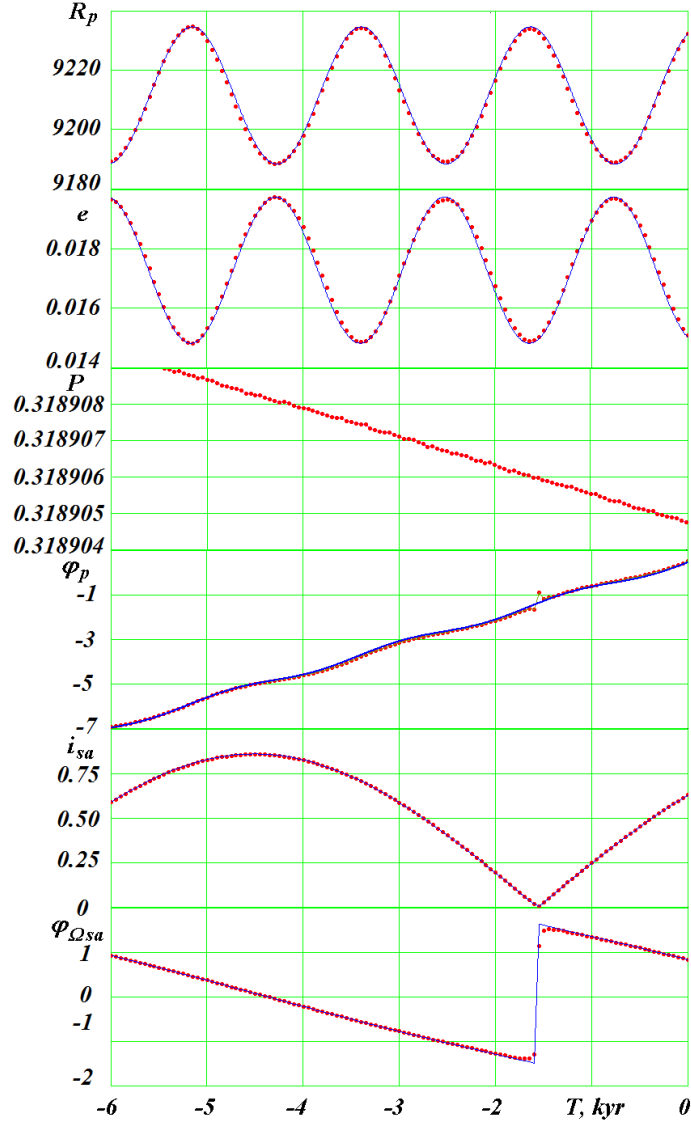


Figure 16. Evolution of Phobos orbit over the last 6 thousand years. The evolution is represented by dots spaced 50 years apart; on the graphs for R_p , e and φ , lines show approximations, while on the lower two graphs, analytical dependencies. For designations, see Fig. 15.

dash-dotted line superimposes the approximating dependence

$$P_{ap}(T) = P_m + P_{A1} \cdot \sin(\varphi_{P01} + 2\pi \cdot T/T_{P1}) + P_{A2} \cdot \sin(\varphi_{P02} + 2\pi \cdot T/T_{P2}), \quad (15)$$

where $P_m = 0.318904778$ day is the average orbital period of Phobos;

$P_{A1} = 7.64 \times 10^{-9}$ day and $P_{A2} = 1.97 \times 10^{-8}$ day are the oscillation amplitudes;

$\varphi_{p01} = 1.005$ and $\varphi_{p02} = -1.885$ are the phases;

$T_{P1} = 2.535$ days and $T_{P2} = 286$ days are the periods of the oscillations.

In expression (15), time T is used in the same units as periods.

As evident from Fig. 15, this biharmonic dependence is in good agreement with the dynamics of variation of the orbital period P .

The pericenter angle φ_p increases into the future with some fluctuations. The angle of orbit inclination i_{sa} and the angle of the ascending node $\varphi_{\Omega sa}$ also vary with some fluctuations, which are not fully manifested in the interval under consideration.

In Fig. 16, the evolution of Phobos orbit over 6 thousand years is shown with dots. The interval between the dots is 50 years. The pericenter radius R_p exhibits stable oscillations with a period of $T_{Rp2} = 1.76$ thousand years and an amplitude of $R_{pA2} = 23.12276$ km. The thin line shows the approximating harmonic

$$R_{pap}(T) = R_{p02} + R_{pA2} \cdot \sin(\varphi_{Rp2} + 2\pi \cdot T/T_{Rp2}), \quad (16)$$

where $R_{p02} = 9211.57$ km and $\varphi_{Rp2} = 1.131$ radians.

The harmonic closely approximates the results represented by the dots. The oscillations occur around the mean value $R_{pm} = 9210.58$ km, which differs from R_{p02} by 1 km.

The results for the eccentricity e can also be well approximated by a harmonic similar to (16) with the same period $T_{e2} = 1.76$ thousand years. In this case, the amplitude is $e_{A2} = 2.466 \times 10^{-3}$, and the values of the remaining parameters are $e_{02} = 1.729 \times 10^{-2}$ and $\varphi_{e2} = -1.963$ radians. The average value of eccentricity $e_m = 1.740 \times 10^{-2}$ also slightly differs from e_{02} .

As shown by the dots in Fig. 16, the orbital period P increases in time into the past at a rate of $P_1 = dP/dT = -2.15 \times 10^{-12}$. This rate is small. It leads to a change in the orbital period of Phobos by unity in the 4th significant digit every 6 thousand years. Solving the equations of interaction of Solar-system bodies with a step of $dT = 1 \times 10^{-5}$ year gives $P_2 = -1.78 \times 10^{-6}$, i.e. 6 orders of magnitude greater rate. Therefore, if we reduce the step down to $dT = 1 \times 10^{-7}$ year, the rate P'_3 will become 6 orders of magnitude lower than P_1 , while the orbital period of Phobos will remain practically unchanged. However, this will lead to an equivalent increase of time required for solving the problem with the same accuracy.

As shown with dots, the pericenter angle φ_p increases into the future with oscillations, i.e., the pericenter rotates counterclockwise in the direction of orbital motion. The approximation shown by the thin line is:

$$\varphi_{pap}(T) = \varphi_{p0} + 2\pi \cdot T/P_{\varphi p} + \varphi_{pA1} \cdot \sin(\varphi_{\varphi p01} + 2\pi \cdot T/T_{\varphi p1}), \quad (17)$$

where $\varphi_{p0} = 0.5121$ radians is the initial value of the pericenter angle;

$P_{\varphi p} = 5.061$ kyr is the pericenter rotation period;

$\varphi_{pA1} = 0.1401$ radians is the oscillation amplitude;

$\varphi_{\varphi p01} = -0.3142$ radians is the phase;

$T_{\varphi p1} = 1.76$ kyr is the pericenter oscillation period.

At the moment the inclination angle passes through zero $i_{sa} = 0$, there is a surge in the magnitude of the angle φ_p . The latter angle is reckoned from the ascending node γ_{sa} of the Phobos orbit (Fig. 14). At the moment the orbital plane $sasa'$ coincides with the equatorial plane A_0A_0 of the Earth, i.e., at $i_{sa} = 0$, the velocity of the ascending node γ_{sa} increases significantly, which circumstance leads to the occurrence of jumps in the graph of the angle φ_p (see Fig. 16). At this time, the pericenter point does not change its motion, i.e., the jumps exerted by the angle φ_p are caused by a sharp increase in the velocity of the node γ_{sa} .

It should be noted that the dependence of the angle φ_p on the ascending-node motion also exists for other planets. Therefore, in a number of our studies [12, 20], when this was important, for example, for Mercury's perihelion, we determined the position of the pericenter from a fixed point on the orbit.

The dots in Fig. 16 show the evolution of the inclination angle i_{sa} and that of the angle $\varphi_{\omega sa}$. Starting from the modern epoch, $T = 0$, the inclination angle i_{sa} , decreases from an initial value of $i_{sa} = 0.630$ radian and at $T = -1.55$ kyr it reaches a value close to zero. Thus, at this epoch the plane of Phobos's orbit $sasa$ turns out to be coincident with the plane of the Earth's equatorial plane A_0A_0 (Fig. 14). The latter plane keeps moving and begins to deviate from the equatorial plane. However, the inclination angles are taken to have positive values, yielding a kink in the graph of the change of i_{sa} .

At the moment the angle i_{sa} passes through zero, the ascending node of the orbit γ_{sa} (Fig. 14) abruptly moves to the opposite end of the diameter, so that the angle $\varphi_{\omega sa}$ changes by π (Fig. 16). After this epoch, the angle i_{sa} increases, reaching a maximum, and then decreases. Over an interval of 6 thousand years, there occurs cycle of change of the angles i_{sa} and $\varphi_{\omega sa}$. It will be shown below that the oscillations of these angles are caused by the precession of the orbital axis $\vec{S}_{sa}$ with a period $T_s = -5.881$ kyr.

In Figs. 15 and 16, we examined the variation of Phobos's orbital parameters over two time intervals. In the third interval, over 80 thousand years into the past, the integration of equations (1) was performed with an output interval of $\Delta T = -5$ kyr, which implies that, here, only 17 data output points were available. Since the largest periods of variation of satellites' orbital parameters are within ΔT , changes in the parameter values over -80 kyr are not representative. They only indicate the limits of oscillations for Phobos orbit; they remain the same as the limits of oscillations over 6 thousand years. Thus, there are no oscillations with longer periods. The results for -80 kyr will

be discussed later when examining the precession of the orbital axis $\vec{S}_{sa}$.

6.3. Dynamics and evolution of Deimos' orbit

The variation of Deimos orbits parameters over 700 continuous revolutions is shown in Fig. 17. As seen from the range of time T , these revolutions last 2.5 years. The pericenter radius R_p is shown with dots that merge into a thick solid line. It performs almost three oscillations with a period $T_{Rp1} = 0.95$ year and an amplitude

$R_{pA1} = 0.109$ km around the average value $R_{pm} = 23450.14$ km. The thin line shows an approximating sinusoid with this period, similar to that contained in formula (16). The initial value of the pericenter radius in this formula is $R_{p0} = R_{pm}$, and the phase is $\varphi_{Rp01} = 0.31416$ radians.

The eccentricity e also varies with this period $T_{e1} = 0.95$ year and the amplitude $e_{A1} = 1.002 \times 10^{-6}$. These oscillations occur around the average eccentricity $e_m = 2.7264 \times 10^{-4}$. The thin line shows a sinusoid with this period and $e_0 = e_m$, $\varphi_{e0} = -2.827$.

As shown in this graph by the thick line, the orbital period P oscillates with two periods around the mean value $P_m = 1.2623993$ days. The thin line shows an approximation similar to (15), in which the parameters have the following values: $P_0 = 1.262399215$ day – initial value;

$P_{A1} = 3.65 \times 10^{-8}$ day and $P_{A2} = 1.77 \times 10^{-6}$ day – oscillation amplitudes;

$\varphi_{p01} = 0$ radians and $\varphi_{p02} = -2.20$ radians – phases;

$T_{P1} = 16.62$ days and $T_{P2} = 345$ days – periods.

Here, time T is used in the same units as the periods.

As seen from Fig. 17, this biharmonic dependence qualitatively reflects the dynamics of the period of Deimos P .

The pericenter angle φ_p increases into the future with some fluctuations. The thin line shows a linear approximation with a pericenter rotation period of 0.626 kyr, which is valid only over this 2.5-year interval.

The orbit inclination angle i_{sa} and the ascending-node angle φ_{wsa} also vary with oscillations that will be discussed in the paragraph on the precession of orbital axes.

In Fig. 18, the evolution of Deimos' orbit over 6 thousand years is shown with dots. The interval between dots is 50 years. The pericenter radius R_p exhibits persistent oscillations with a period $T_{Rp2} = 0.463$ thousand years and an amplitude $R_{pA2} = 1.193$ km. The thin line shows the approximating harmonic (16), in which $R_{p02} = R_{pm} = 23448.99$ km, and $\varphi_{Rp2} = 2.042$ radians.

The harmonic well approximates the results represented by points only in the central region of the oscillations. However, the extreme values of the oscillations may not coincide with the extremes of the harmonic.

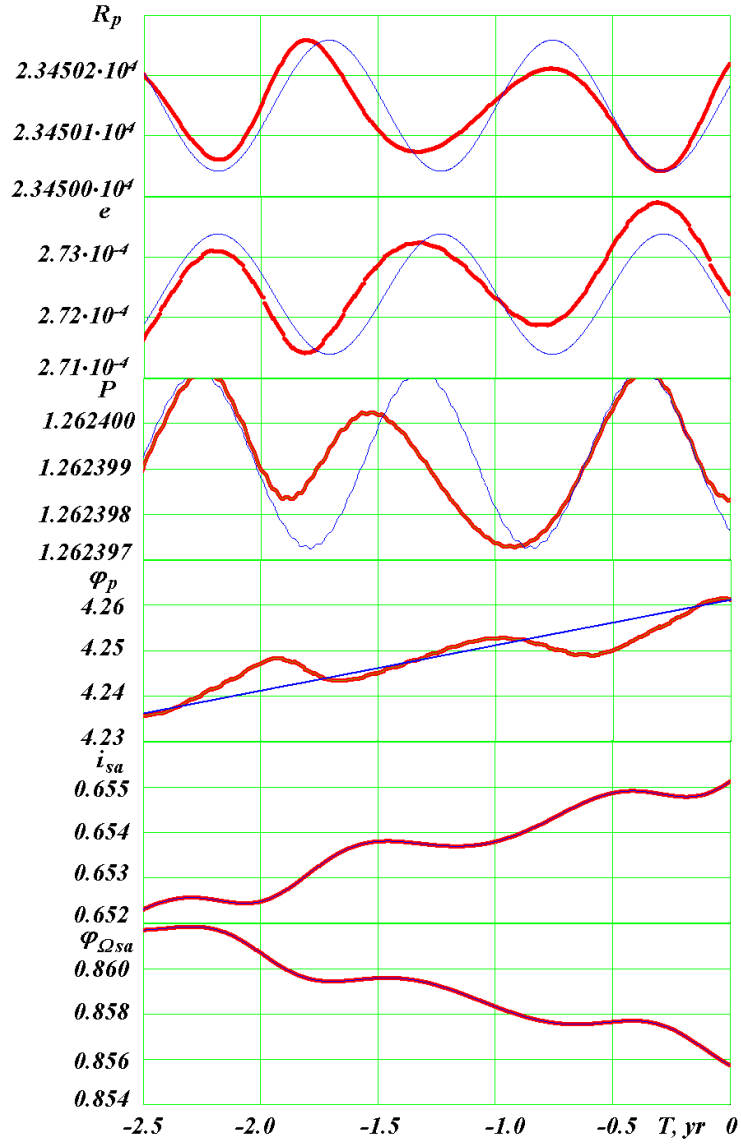


Figure 17. Dynamics of the orbital elements of Deimos over 700 continuous its revolutions. The thin lines in the three upper graphs show the approximating sinusoids. The graph of φ_p is shown with a superimposed linear approximation. For other designations, see Fig. 15.

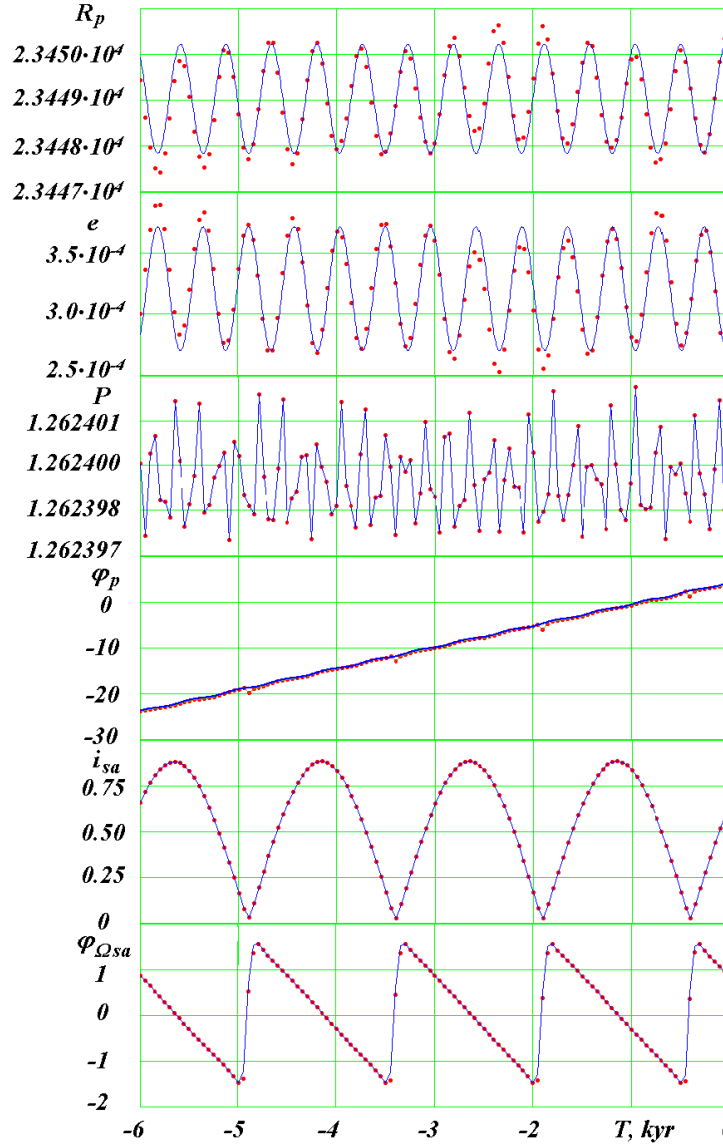


Figure 18. Evolution of Deimos' orbit over the last 6 thousand years. Except for the period P , the thin lines in the graphs show approximating relations, and the two lower graphs show analytical relations. For other designations, see Fig. 15.

With the same difference, the results for eccentricity e are also approximated by a harmonic similar to that in formula (16) with the same

period $T_{e2} = 0.463$ thousand years. In this case, the amplitude is $e_{A2} = 5.067 \times 10^{-5}$, and the remaining parameters are $e_{02} = 3.211 \times 10^{-4}$ and $\varphi_{e2} = -1.10$ radians. The average value of eccentricity is $e_m = 3.223 \times 10^{-4}$.

As shown by the dots in Fig. 18, the period of revolution P fluctuates within the same limits as in the interval of 700 revolutions in Fig. 17. Therefore, there are no oscillations with a large period, and there is no tendency for the period P to increase. Thus, the period of revolution of Deimos fluctuates with the periods $P_{A1} = 3.65 \times 10^{-8}$ day and $P_{A2} = 1.77 \times 10^{-6}$ day established above.

As shown by the dots, the pericenter angle φ_p increases into the future with oscillations, i.e. the pericenter rotates counterclockwise in the direction of orbital motion. The approximation shown by the thin line has the form (17), where $\varphi_{p0} = 4.246$ radians is the initial value of the pericenter angle;

$P_{\varphi p} = 1.3428$ kyr – pericenter rotation period;

$\varphi_{pA1} = 0.1662$ radians – oscillation amplitude;

$\varphi_{\varphi p01} = 0.6283$ radians – phase;

$T_{\varphi p1} = 0.463$ kyr – pericenter oscillation period.

Approximation (17) agrees well with the change of the pericenter angle with the exception of the jumps of φ_p that occur during the epochs of coincidence of the plane of Deimos orbit with the Earth's equator, i.e., when $i_{sa} = 0$. As already discussed for Phobos, these jumps relate to the reference point γ_{sa} in Fig. 14, and the pericenter angle φ_p varies without jumps, as shown by the thin line in Fig. 18.

The dots in Fig. 18 show the evolution of the inclination angle i_{sa} and the angle $\varphi_{\Omega sa}$. The inclination angle i_{sa} undergoes 5 regular oscillations over this interval from zero to a maximum value of $i_{samax} = 0.885$. The angle $\varphi_{\omega sa}$ undergoes the same number of oscillations. Their period is $T_i = T_{\varphi\Omega} = -T_S = 1.501$ kyr.

In Figs. 17 and 18, we examined the variation of Deimos' orbital parameters over two time intervals. In the third interval, over the past 80 thousand years, as already noted for Phobos, the parameter changes over -80 kyr are not representative. They merely indicate the limits of oscillation. Like for Phobos, for Deimos they remain the same as the limits of oscillation over 6 thousand years. Thus, there are no oscillations with longer periods. The results for the time interval of -80 kyr will be discussed below when examining the precession of the orbital axis $\vec{S}_{sa}$.

6.4. Precession of the orbital axes of Mars' satellites

6.4.1. Specific features of the precession of the Moon orbital axis

Projections S_{sax} , S_{say} and S_{saz} of a satellite's orbital axis $\vec{S}_{sa}$ in the equatorial coordinate system xyz (Fig. 14) are defined by the angles specifying the

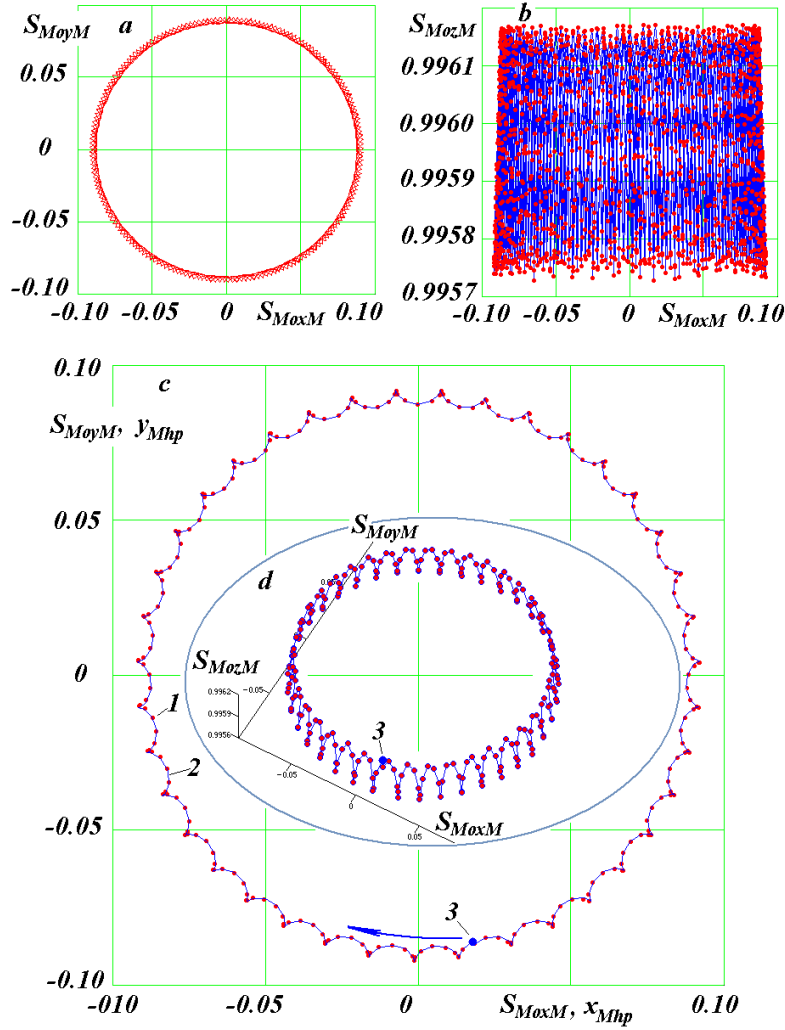


Figure 19. Projections of the precession of the Moon's orbital axis $\vec{S}_{Mo}$ for the spans of 113.4 (a, b) and 18.6 years from $T = 0$ (the epoch of December 30, 1949) to $T = 0.186$ (c, d) onto the axes of the coordinate system $x_M y_M z_M$: d – in three-dimensional form; point 1 – projection of the axis onto the $y_M x_M$ plane; line 2 – equation of the hypocycloid $y_{Mhp}(x_{Mhp})$ (19) - (20); 3 – position of the axis at $T = 0$.

position of the orbital plane of the satellite, i_{sa} and $\varphi_{\Omega sa}$, and can be calculated using the same algorithm (7) that was used for planets [32].

Projections of the axis $\vec{S}_{sa}$ onto the axes of the coordinate system $x_M y_M z_M$ (Fig. 14) are calculated similarly to equations (9) but with the angles i and φ_Ω , which define the position of the orbital plane of the planet, used instead of the angles i_M and φ_M that define the position of Laplace plane.

The angle of inclination of the satellite's orbital plane θ_{sa} and the precession angle ψ_{sa} relative to the planet's moving orbit can be determined using respectively expressions (9) and (10).

Smulsky (2022) [32] examined in detail the evolution of the Moon's orbit over the period of 100 million years and found that the Moon orbital axis $\vec{S}_{Mo} = \vec{S}_{sa}$ (Fig. 14) precesses around the Earth's orbital axis $\vec{S}_E = \vec{S}_{pl}$. Several properties of this precession were also discovered that are not characteristic of planets. Those properties are also inherent to the precessions of the orbital axes of Mars' satellites. Therefore, we will first consider the results of [32] on the precession of the Moon orbital axis.

Figure 19a shows the motion of the end of the orbital axis $\vec{S}_{Mo}$ projected onto the $y_M x_M$ plane over the 113.4-year time interval under study, during which $2 \times 736 = 1472$ continuous revolutions of the Moon occur. The graph shows that the end of the vector $\vec{S}_{Mo}$ moves in a circle with slight oscillations. The orbital period is $T_S = 18.6006$ years, and the period of oscillation of the inclination angle θ_{sa} , or the precession angle ψ_{sa} , (Fig. 14) is $T_{\psi 1} = 0.4745$ years. Over the time interval under consideration, the axis $\vec{S}_{Mo}$ makes 6 clockwise revolutions.

From the projection onto the $z_M x_M$ plane (Fig. 19b) it is evident that the end of the vector $\vec{S}_{Mo}$ performs small oscillations along the z_M -axis with a swing of $\Delta z_M = 4.43 \times 10^{-4}$. Those oscillations are symmetrical with respect to the x_M -axis. Thus, the Moon orbital axis precesses in clockwise direction relative to the moving Earth orbital axis $\vec{S}_E$ with a precession period $T_S = 18.6006$ years and nutational oscillations with a period $T_{\theta 1} = T_{\psi 1} = 0.4745$ year. The amplitude of the oscillation of the precession angle is $\Delta \psi_{MoA} = 0.023662$ radian, and the inclination angle oscillates with an amplitude of $\theta_{MoA} = 0.002464 = 8.4692'$ around the mean value $\theta_{Mom} = 0.09006 = 5.1544^\circ$.

Figure 19c shows, at a larger scale, the projection onto the $y_M x_M$ -plane of the precession of the orbital axis $\vec{S}_{Mo}$ over one precession period from $T = 0$ to $T = 0.186$, i.e., over 18.6 years. Beginning from the time $T = 0$, marked with point 3, the end of the axis moves clockwise. The graph shows that the end of the vector $\vec{S}_{Mo}$ moves exactly around a circle, and the nutational oscillations are regular. In Fig. 19d, the dots and the line show the precession of the orbital axis $\vec{S}_{Mo}$ in three dimensions. In the graph, the scale along the vertical z_M axis is increased by a factor of 40.

The analysis revealed that the end of the vector $\vec{S}_{Mo}$ moves exactly along

a hypocycloid. A hypocycloid is formed by a point on a circle of radius r , rolling, in this case, in clockwise direction without slipping along the inside of another circle of radius R . The first circle, meanwhile, rotates counterclockwise.

On the $y_M x_M$ -plane (Fig. 19c), as the radius of the large circle $R = \sin \theta_{Mom}$, the mean value of the projection of the orbital axis $\vec{S}_{Mo}$ onto this plane is to be chosen, with the radius of the small circle $r = \sin \theta_{MoA}$ being the amplitude of its oscillations. The center of the small circle is translated with angular velocity $2\pi/T_S$ in clockwise direction. In this translational motion, the vector $\vec{S}_{Mo}$ oscillates with a period $T_{\psi 1}$, i.e. with an angular velocity of $2\pi/T_{\psi 1}$. Then the absolute angular velocity of rotation of the small circle $2\pi/T_n$ will be equal to the sum: $2\pi/T_n = 2\pi/T_{\psi 1} + 2\pi/T_S$. Therefore, the period of nutational rotation will be

$$T_n = \frac{T_S \cdot T_{\psi 1}}{T_S + T_{\psi 1}} = 0.48692 \text{ year.} \quad (18)$$

Then, the equation of the hypocycloid on the $y_M x_M$ -plane will be as follows:

$$x_{Mhp} = R \cos \left(\varphi_{10} + 2\pi \frac{T}{T_S} \right) + r \cos \left(\varphi_{20} + 2\pi \frac{T}{T_n} \right); \quad (19)$$

$$y_{Mhp} = R \sin \left(\varphi_{10} + 2\pi \frac{T}{T_S} \right) + r \sin \left(\varphi_{20} + 2\pi \frac{T}{T_n} \right), \quad (20)$$

where $\varphi_{10} = 4.92766$ radians and $\varphi_{20} = 2.19315$ radians are the initial phases that determine the positions of the vector $\vec{S}_{Mo}$ on the circles at the initial time $T = 0$.

In Fig. 19c, the line shows the trajectory of motion along the hypocycloid (19) - (20), and the dots show the projection of the motion of the Moon orbital axis $\vec{S}_{Mo}$. Evidently, the line and the dots are perfectly coincident. Thus, the axis $\vec{S}_{Mo}$ of the Moon orbit executes an averaged motion around the axis $\vec{S}_E$ of the Earth's orbit in clockwise direction with a period T_S and an average angle $\theta_{Mom} = 5.1544^\circ$. Around the averaged motion, the axis $\vec{S}_{Mo}$ of the orbit executes a second rotational motion in counterclockwise direction with a period T_n and an angular deviation $\theta_{MoA} = 8.4692'$ from the average axis. During a complete revolution of the averaged axis of the Moon's orbit, $-T_S/T_\psi = 39.2$, nutational revolutions of the instantaneous axis $\vec{S}_{Mo}$ occur.

6.4.2. Specific features of the orbital precession of Mars satellites

Figure 20a shows the precession of the orbital axis of Phobos (Ph) $\vec{S}_{sa}$ in projection onto the $y_M x_M$ -plane, which is perpendicular to the Mars orbital

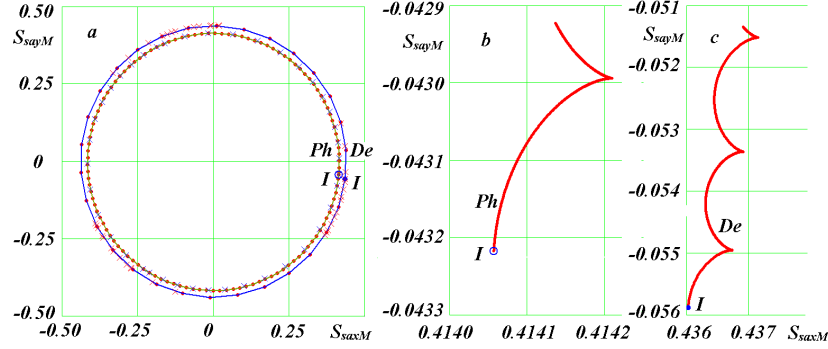


Figure 20. Precessions of the orbital axes $\vec{S}_{sa}$ of Mars satellites, Phobos (Ph) and Deimos (De), around the axis $\vec{S}_{pl}$ of the mobile orbit of Mars: *a* for full periods of precession T_S ; the interval between dots is 50 years; *I* the initial position of the orbital axes in the epoch of 2000.0; xs show data for 80 thousand years; *b* and *c* sections of the precession of the orbital axes for 700 continuous revolutions of the satellites on an enlarged scale.

axis $\vec{S}_{pl}$ (Fig. 14) over a full precession period $T_S = -5.8816$ kyr. The projections of the end of the axis are $\vec{S}_{sa}$ shown with dots following with an interval of 50 years.

The position of the end of the orbital axis $\vec{S}_{sa}$ at the initial time (Fig. 20a) is shown with the large dot *I*. From this dot on, the axis moves counterclockwise into the past and clockwise into the future. This motion is symmetrical about the origin, onto which the Mars orbital axis is $\vec{S}_{sa}$ projected. Thus, the satellite's orbital axis $\vec{S}_{sa}$ rotates clockwise around the axis $\vec{S}_{pl}$, i.e., in the direction opposite to the orbital motion of Phobos around Mars.

Over the time interval of 6 thousand years, the inclination angle θ_s varies in the range $0.4268 < \theta_{sa} < 0.4312$ radians with an average value $\theta_{sam} = 0.4289$ radians = 24.5765° . The angles θ_{sa} and ψ_{sa} exhibit oscillations with two periods, $T_{\theta 1} = 0.27$ kyr and $T_{\theta 2} = 6.175$ kyr, and with the following amplitudes: $\theta_{A1} = 1.130 \times 10^{-4}$, $\psi_{A1} = 3.023 \times 10^{-4}$, $\theta_{A2} = 2.10 \times 10^{-3}$ and $\psi_{A2} = 5.573 \times 10^{-3}$ radian, respectively. The amplitudes of the oscillations are small compared to the average radius of the precession circle $R = \sin \theta_{sam}$; that is why those oscillations of the axis in Fig. 20a are not visible.

Figure 20b shows the precession of the Phobos orbital axis $\vec{S}_{sa}(Ph)$ on a larger scale, drawn from the data taken from 700 orbital revolutions. A single nutational motion is visible here, from the initial position of the axis *I* to the position at the peak apex. Thus, like the Moon orbital axis, the Phobos orbital axis performs a mean clockwise motion around a circle of radius R and counterclockwise nutational rotations around a very small circle of radius

$r = \sin \theta_{A1}$. However, the Moon's nutational motions are fixed, while the Phobos nutational motions are variable. They are modulated by the second period $T_{\theta 2}$, i.e., the radius of the precession circle r will change during the precession cycle T_S .

The precession of the orbital axis of Demos (De), shown in Fig. 20a has a slightly larger precession radius R . The precession period is $T_S = -1.5015$ kyr, i.e. it is 4 times shorter than the precession period of the orbital axis of Phobos.

Over 6 thousand years, the inclination angle of the Deimos (De) orbital axis θ_{sa} varies within the range of $0.4534 < \theta_{sa} < 0.4557$ radians with an average value $\theta_{sam} = 0.4545$ radians $= 26.0418^\circ$. The angles θ_{sa} and ψ_{sa} also exhibit oscillations with two periods $T_{\theta 1} = 0.21298$ kyr and $T_{\theta 2} = 2.17$ kyr and the following amplitudes: $\theta_{A1} = 3.434 \times 10^{-4}$, $\psi_{A1} = 5.901 \times 10^{-4}$, $\theta_{A2} = 7.693 \times 10^{-4}$ and $\psi_{A2} = 2.213 \times 10^{-3}$ radian, respectively. The amplitudes of the oscillations are also small compared to the average radius of the precession circle $R = \sin \theta_{sam}$, so these oscillations of the axis are not visible in Fig. 20a.

Figure 20c shows the precession of Deimos' orbital axis $\vec{S}_{sa}(De)$ on a larger scale based on the data taken from 700 orbital revolutions. Here, from the initial position of the axis I , nearly three nutational motions occur, with variations in their duration being visible. Thus, like in the case of Phobos, the Deimos orbital axis performs a clockwise mean motion around a circle of radius R and counterclockwise nutational rotations around a very small circle of radius $r = \sin \theta_{A1}$. These nutational motions, like those of Phobos, are variable. They are modulated by the second period $T_{\theta 2}$, so that the radius of the precessional circle r changes during the precession cycle T_S .

In Fig. 20a, results for -80 kyr are plotted with x's on the precession circles of the Phobos and Deimos orbital axes. During this interval, the orbital axes complete tens of revolutions. Since the output interval $\Delta T = 5$ kyr is not a multiple of T_S , these results are unevenly distributed on the precession circles. However, they fit them precisely. Thus, the results obtained for -80 kyr are completely consistent with the results for -6 kyr.

6.4.3. Evolution of the precession and inclination angles of Mars' satellites

Changes in the precession angle ψ_{sa} and inclination angle θ_{sa} of Phobos' orbit (Ph) are shown with the dots in Fig. 21 for the period of the last 6 thousand years. The angle ψ_{sa} decreases almost linearly into the future, i.e., the axis rotates clockwise. As already noted, the precession period of the axis is $T_S = -5.8816$ kyr. Due to the smallness of the oscillations of the angle ψ_{sa} , those oscillations are not visible on the graph. The oscillations were determined considering the difference between ψ_{sa} and its linear trend. There exist two oscillation periods $T_{\theta 1}$ and $T_{\theta 2}$. With this fact taken into account,

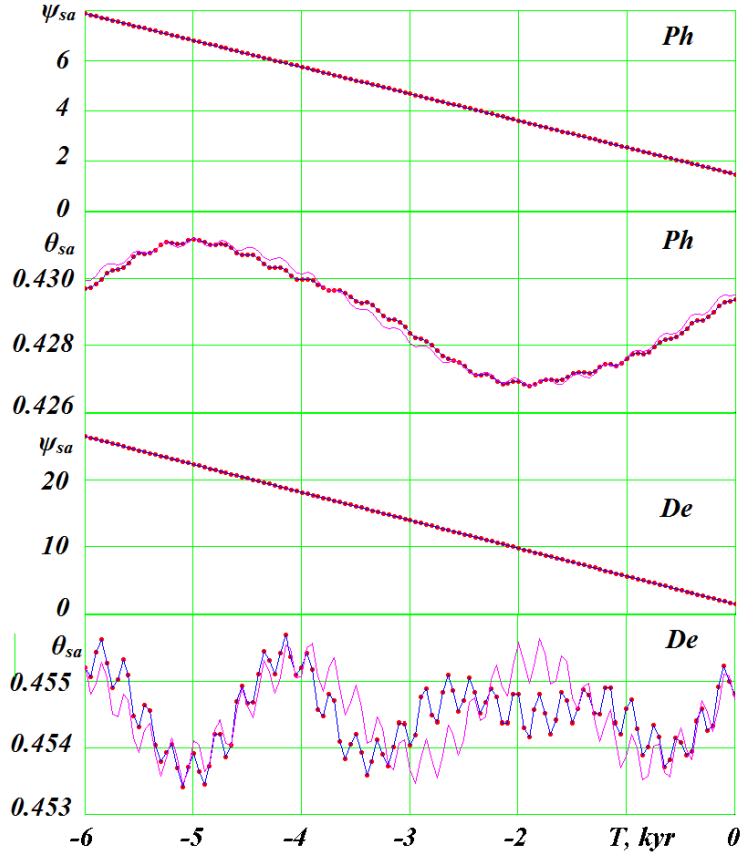


Figure 21. Evolution of the inclination and precession angles (θ_{sa} and ψ_{sa}) of the orbits of Mars' satellites, Phobos (Ph) and Deimos (De), relative to the moving axis $\vec{S}_{pl}$ of the Mars' orbit over the last 6 thousand years. The thin lines on the graphs show the corresponding approximate dependences.

the approximation is as follows:

$$\psi_{saap}(T) = \psi_{sa0} + 2\pi \cdot T/T_S + \psi_{A1} \cdot \sin(\varphi_{\psi 1} + 2\pi \cdot T/T_{\theta 1}) + \psi_{A2} \cdot \sin(\varphi_{\psi 2} + 2\pi \cdot T/T_{\theta 2}) \quad (21)$$

where $\psi_{sa0} = 1.47203$ radians; $\varphi_{\psi 1} = 2.985$ radians; $\varphi_{\psi 2} = -1.382$ radians.

In Fig. 21, dependence (21) is shown with a thin line. The angle θ_{sa} on this interval performs one oscillation with the period $T_{\theta 2}$. Smaller oscillations with the period $T_{\theta 1}$ are also visible on the graph. The thin line shows the

approximation

$$\theta_{saap}(T) = \theta_{sa0} + \theta_{A1} \cdot \sin(\varphi_{\theta1} + 2\pi \cdot T/T_{\theta1}) + \theta_{A2} \cdot \sin(\varphi_{\theta2} + 2\pi \cdot T/T_{\theta2}) \quad (22)$$

with $\theta_{sa0} = 0.428989$ radians; $\varphi_{\theta1} = -1.571$ radians; $\varphi_{\theta2} = 0.314$ radians.

The approximation line (22) does not exactly coincide with the points, but it well reflects the periods and amplitudes of the oscillations.

From the angles ψ_{sa} and θ_{sa} , we can determine the angles i_{sa} and $\varphi_{\Omega sa}$ (Fig. 14) that define the position of the Phobos orbital plane with respect to the equatorial coordinate system xyz [32]. These transformations are similar to transformations (13) - (14) obtained earlier for planets, so we will not present them. For applying the transformations to satellites, it is necessary to replace the Laplace-plane angles i_M and φ_M with the planets plane angles i and φ_Ω . In doing so, it is necessary to take into account the fact that the angles i_M and φ_M have fixed values while the angles i and φ_Ω change over time. Into the dependences for Phobos similar to (13) - (14) the approximations $\psi_{saap}(T)$ and θ_{saap} were substituted. In Fig. 16, those dependences are shown as thin lines for the angles i_{sa} and $\varphi_{\Omega sa}$. Evidently, approximations (21) - (22) well reflect changes in the angles of the Phobos orbital plane. Similar comparisons were performed for the changes in these parameters over the -80 kyr interval as well. The data also matched. Therefore, the above approximations can be used over any time interval.

The variation of the same precession and inclination angles ψ_{sa} and θ_{sa} of the Deimos (De) orbit are also shown with the dots in Fig. 21 for the last 6 thousand years. The angle ψ_{sa} decreases into the future almost linearly with the period of precession of the axis $T_S = -1.5015$ kyr. There are also two oscillation periods $T_{\theta1}$ and $T_{\theta2}$, which are not visible on the graph due to the smallness of the oscillations of the angle ψ_{sa} . The thin line shows the approximation (21), in which $\psi_{sa0} = 1.44929$ radians; $\varphi_{\psi1} = 3.142$ radians, and $\varphi_{\psi2} = -0.942$ radians.

The angle θ_{sa} of inclination of the Deimos orbital axis (De) undergoes several oscillations during this interval, with both the first and second periods. Oscillations with the longer period are irregular. Approximation (22) is shown with a thin line, in which the parameters have the following values: $\theta_{sa0} = 0.454548$ radians; $\varphi_{\theta1} = -2.199$ radians; and $\varphi_{\theta2} = 0.628$ radians. The oscillations of the approximation with the shorter period are in better agreement with the dots. The oscillations of the approximation with the longer period differ from the oscillations of the inclination angle θ_{sa} during the moments of irregularity of this angle. Nevertheless, approximation (22) reflects the behavior of the angle θ_s and the limits of its change.

For Deimos, the angles i_{sa} and $\varphi_{\omega sa}$ that characterize the position of the Deimos orbital plane relative to the equatorial coordinate system xyz were also determined from the approximations $\psi_{saap}(T)$ and $\theta_{saap}(T)$ using the dependencies similar to (13) - (14). In Fig. 18, the results are shown as thin

lines. Despite the differences noted above, the approximations (21) - (22) rather accurately describe the evolution of the Deimos orbital plane. This is due to the fact that the amplitudes of oscillations of the angles θ_{A1} and θ_{A2} for example, for the angle θ_{sa} are significantly smaller than the amplitudes of oscillations of the angles i_{sa} and $\varphi_{\omega sa}$.

Similar comparisons were also made for these parameters over the -80 kyr interval. The data also matched. Therefore, given the differences noted, those approximations can also be used over any time interval.

6.4.4. *Mathematical models of satellite motion*

As noted, Smulsky (2022) [32] presented a mathematical model of the Moon based on six orbital parameters: R_p , e , φ_p , P , i_{Mo} , $\varphi_{\Omega Mo}$. Analytical time dependences of these parameters for the orbits of Mars' satellites are presented above. Their degree of correspondence to actual parameters is demonstrated. With these parameters, the algorithm of the mathematical model for the Moon can be applied to the satellites of Mars.

For the Moon, the parameter approximations accurately describe its short-period oscillations. For Phobos, there are some differences, which in some cases become noticeable for Deimos, as for θ_{sa} in Fig. 21. However, these differences concern very short oscillations, and their amplitudes are also very small. Therefore, taking these differences into account, mathematical models of Phobos and Deimos' motion can be used over unlimited time intervals.

7. Conclusion

Table 2 provides some parameters for the evolution of the orbits of Mars, minor planets, and its satellites for comparison. The mean semi-axes a_m of the minor planets are 1.6-1.8 times larger, and their orbital periods P_{orm} are 1.9-2.4 times larger than the corresponding parameters of Mars' orbit. However, the orbital precession periods T_S and perihelion rotation periods T_p , as well as the oscillation periods of the orbital elements Te_1 and Te_2 , are several times smaller than those of Mars' orbit.

It should be noted that for all planets, as well as for Mars, those periods are considerably higher than for minor planets [8, 16].

For satellites, the above-mentioned periods are several times shorter than those for minor planets, and the oscillation amplitudes are several orders of magnitude smaller. The amplitudes of eccentricity oscillation for Deimos are especially small.

Among the planets, Pallas has the greatest eccentricity, and Mars the least. The orbits of the satellites are slightly eccentric, and the orbit of Deimos is almost circular.

Table 2. Some parameters of the orbital evolution of Mars, minor planets, and its satellites: a_m is the average semi-major axis of the orbit in astronomical units; R_{pm} is the average pericenter radius in km; P_{orm} is the average sidereal period of revolution in sidereal Earth years, and P_{ormd} is the average sidereal period of revolution in days; Te_1 and Te_2 are the first and second periods of eccentricity oscillations in kyr; e_{A1} and e_{A2} are the amplitudes of eccentricity oscillations with the first and second periods; e_m is the average orbital eccentricity; T_s is the precession period of the orbital axis in kyr; $\varphi_{\omega m}$ is the average angle of the ascending node of the orbit; i_m is the average angle of inclination of the orbital plane to the plane of fixed Earth equator; θ_{sam} is the average angle of inclination of the orbital planes of a satellite and its planet; T_p is the average period of revolution of the pericenter of the orbit in kyr.

** - the value $\varphi_{\Omega m} = 0.043$ for Pallas differs from the value 0.067 due to the small number of dots with a large range of $\varphi_{\Omega m}$.

Parameters	Planets				Satellites	
	Mars	Ceres	Pallas	Vesta	Phobos	Deimos
a_m/R_{pm}^*	1.523687	2.76738	2.77056	2.36153	9210.58*	23448.99*
P_{orm}/P_{ormd}^*	1.880815	4.60393	4.61166	3.62912	0.3189047788*	1.2623993*
Te_1	95	26	14.3	39.9	0.0009	0.00095
e_{A1}	0.024	0.043	0.14	0.036	$1.4 \cdot 10^{-5}$	$1.002 \cdot 10^{-6}$
Te_2	$2.32 \cdot 10^3$	640	231	150	1.76	0.463
e_{A2}	0.032	0.0097	0.005	0.030	$2.466 \cdot 10^{-2}$	$5.067 \cdot 10^{-5}$
e_m	0.066	0.118	0.287	0.104	0.0174	$3.223 \cdot 10^{-4}$
T_s	-73	-21.9	-27.9	-32.7	-5.881	-1.501
$\varphi_{\Omega m}$	0.0678018	0.067	0.043**	0.067	- -	- -
i_m/θ_{sam}^*	0.405044	0.419	0.644	0.409	0.4289*	0.4545*
T_p	77.5	23.9	28.7	35.1	5.061	1.3428

The inclination angle (i_m) of the planetary orbital planes is greatest for Pallas and smallest for Mars. The angles of inclination (θ_{sam}) of the satellites' orbital planes to Mars' orbital plane are similar in magnitude, but are 4-5 times greater than the angle of inclination of the Moon's orbit to Earth's orbit.

The obtained orbital parameters for the bodies examined in the contemporary epoch are consistent with those determined from observations. This ensures that the calculated parameters for very short and very long orbital variations, which cannot be obtained from observations, indeed determine the dynamics and evolution of the bodies examined.

As a result of the performed research and comparisons, it was established that the minor planets and Mars' satellites exert no noticeable influence on the evolution of Mars' orbit.

The orbits of the examined bodies change as a result of four motions. The

orbit is deformed in its plane, i.e., its eccentricity changes. The orbit rotates in its plane in the direction of its orbital motion, i.e., its pericenter rotates. The axes of the planets' orbits rotate counter to their orbital motion relative to the angular momentum of the entire Solar system and, in addition, oscillate. The last two motions, the third and fourth ones, occur in the satellites of the planets relative to the axes of their planets' moving orbits.

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A Appendix

Table A1. The masses m_{bj} and radii Ra_j of the planets from Mercury to Pluto, the Moon, the Sun ($j = 1 - 11$); minor planets: Ceres, Pallas and Vesta (12 – 14) and satellites of Mars: Phobos and Deimos (15 – 16) and the initial condition on epoch 2000.0 $JD_s = 2451545$ in the heliocentric equatorial coordinate system on epoch 2000.0.

Bodies,	Bodies masses in kg, their radii and coordinates in m and velocities in m/s			
j	m_{bj}, Ra_j	x_{aj}, v_{xaj}	y_{aj}, v_{yaj}	z_{aj}, v_{zaj}
1	3.30187977984878E+23	-19462339864.0595	-59927846489.3697	-29992647261.237
	2439700.	36994.914171908	-8529.87885527349	-8393.22191793989
2	4.86855317055688E+24	-107456496126.607	-6922534453.29301	3686184808.17613
	6051800.	1381.90740360098	-32017.8185431599	-14491.8355763523
3	5.97369891237542E+24	-26499043506.3049	132757415909.924	57556718057.6781
	6378137.	-29794.2597404869	-5018.0534033582	-2175.39391044519
4	6.41855339747085E+23	208048148450.173	209652454.426534	-5529147651.64835
	3396190.	1162.66970292897	23918.4093508882	10939.1717835323
5	1.89900420678135E+27	598567582123.927	409386370668.236	160894307218.866
	71492000.	-7909.83774351073	10183.4980244277	4557.71855630437
6	5.68603573730455E+26	958385123854.86	923715662042.261	340300867450.862
	60268000.	-7432.02200648945	6109.17631441914	2842.9335617929
7	8.68410707095472E+25	2158975002671.38	-1870910980891.57	-849968367371.141
	25555900.	4637.02341657522	4262.8547370183	1801.4227322724
8	1.0245699196367E+26	2515046649989.58	-3437774681251.1	-1469713260290.85
	24764000.	4465.9023757439	2888.80492159027	1071.4511263207
9	1.46419905915066E+22	-1477331954448.78	-4185579228356.77	-860739893790.863
	1195000.	5244.15154693681	-1952.37742980271	-2190.51718451347
10	7.34767167823222E+22	-26790671953.8239	132490722789.693	57480601645.3155
	1737400.	-29150.7748940034	-5684.16365194957	-2476.75429757996
11	1.98891944570663E+30	0	0	0
	697113000.	0	0	0
12	9.3242807151709E+20	-355942355024.543	81631220918.0452	110885727106.563
	474000.	-6205.93653700676	-17046.5696297186	-6760.54952625679
13	2.00582283208438E+20	-125832530260.662	290884815303.125	-48986495808.8365
	266000.	-20323.0355355139	-9177.30472275436	3225.3987386901
14	2.58772626846622E+20	-202492724814.64	-242431717996.622	-70066706651.5563
	265000.	16664.9344399658	-11108.5030384489	-6600.84498709272
15	1.08E+16	208046158986.227	200909419.452672	-5532329600.09171
	11071.	3005.87707332432	23875.3044205087	9920.83979227317
16	1.80 E+15	208058512751.238	193904112.180326	-5543093640.62256
	6250.	2203.5936897355	24761.8594386888	10760.2324693391