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Multilayer Ring Structures

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Abstract—A study of the behavior of axisymmetric structures is important for understanding the problem of the existence and stability of planet rings, spherical star constellations, and galaxies. The multilayer ring structure algorithm is developed on the basis of an exact solution to the problem of n-body gravitational axisymmetric interaction. As a result of the numerical integration of differential motion equations of point bodies composing the above structures, the evolution of several of their models is investigated. Some of them are invariable in configuration, others change forms due to interlayer interactions, and the rest throw part of bodies out of the structure.

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Problems of the existence and stability of certain configurations of material particles mutually gravitating according to Newton's law have been studied in [1–7]. These configurations are axisymmetric structures of four, five, six, seven, eight, nine, ten, and more bodies. B. Elmabsout and E.A. Grebenikov in their models [5, 6] proved the existence of axisymmetic strictures with an arbitrary number of components under conditions of the fulfillment of certain exact analytical conditions of gravitational (masses) and geometric model parameters (the distance between bodies and the angular rate of motion around a common center).

In our opinion, the use of the above results along with the ring structure methods [8] is of interest for studying the problem of the existence and stability of planet rings, spherical star constellations, and galaxies. Such an approach is based on an exact solution of the n-body axisymmetric interaction problem [9].

Each peripheral body, e.g., $B_{1,1}$ (Fig. 1) of m_1 in mass is affected by other peripheral bodies $B_{1,1}$, $B_{1,2}$, … $B_{1,n1}$ of a given ring and the central body B_0 with a centrally directed force [9]

$$F = G(m_0 + m_1 f_n)m_1/r^2, \tag{1}$$

where G is the gravitation constant and r is the distance from the central body m_0 to the peripheral one m_1; the function f_n depends on the number of bodies n:

$$f_n = 0.25\sum_{i=2}^{n}\frac{1}{\sin[(i-1)\pi/n]}. \tag{2}$$

Under the action of force (1), all peripheral bodies move along a trajectory, which can be shown as

$$r = \frac{R_p}{(\alpha_1 + 1)\cos(\varphi - \varphi_p) - \alpha_1}, \tag{3}$$

in the polar coordinates (r, φ), where the angle φ is measured from the x axis, φ_p is the pericenter angle, $\alpha_1 = \mu_1/(R_p v_p^2)$ is the trajectory parameter, $\mu_1 = -G(m_0 + m_1 f_n)$ is the interaction parameter, R_p is the pericenter radius, and v_p is the body velocity in the pericenter.

Equation (3) is a circle at $\alpha_1 = -1$, ellipse at $-1 < \alpha_1 < -0.5$, parabola at $\alpha_1 = -0.5$, and hyperbola at

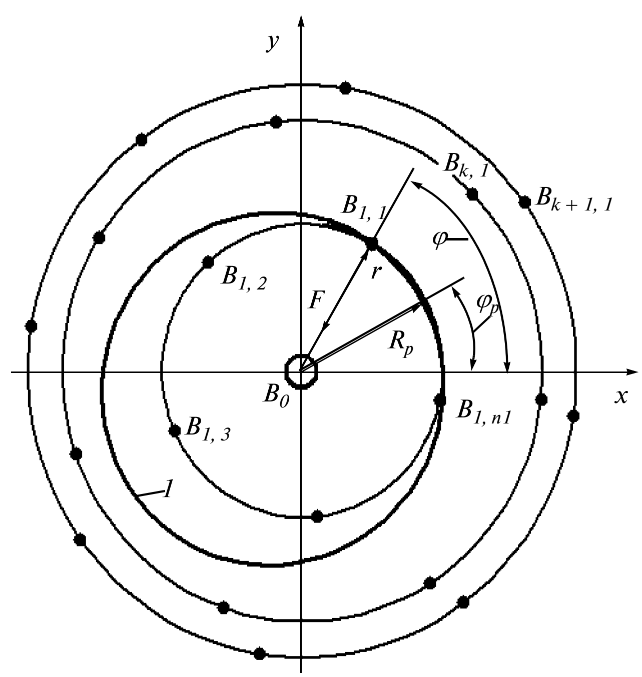

Fig. 1. Ring structure of peripheral bodies $B_{k,\,ik}$ axisymmetrized relative to the central body B_0; the numeric *1* shows the elliptical orbit of the first body on the first ring.

$-0.5 < \alpha_1 < 0$. Let us also write the equations for the radial v_r and transversal v_t velocities:

$$\begin{aligned} v_r &= \pm v_p\sqrt{(\alpha_1+1)^2-(\alpha_1+R_p/r)^2}, \\ v_t &= v_pR_p/r, \end{aligned} \tag{4}$$

where $v_r < 0$ when moving from the apocenter to the pericenter, i.e., at $2\pi > \varphi - \varphi_p > \pi$. The equations for the eccentricity of orbit e, its semimajor axis a, and the orbital period T are

$$\begin{aligned} e &= -(1+1/\alpha_1), \quad a = \frac{2R_p\alpha_1}{2\alpha_1-1}; \\ T &= \frac{2\pi a^{3/2}}{\sqrt{-\mu_1}}. \end{aligned} \tag{5}$$

If the body $B_{1,1}$ has the polar coordinate r_0 and φ_0 at the initial time (Fig. 1), then its Cartesian coordinates and in-plane velocities can be written as

$$x_0 = r_0\cos\varphi_0, \quad y_0 = r_0\sin\varphi_0, \tag{6}$$

$$\begin{aligned} v_{x0} &= v_r\cos\varphi_0 - v_t\sin\varphi_0, \\ v_{y0} &= v_r\sin\varphi_0 + v_t\cos\varphi_0. \end{aligned} \tag{7}$$

The construction of ring structures is based on the following two principles.

(1) The action force on a body out of a ring structure is equal to the force induced by a body in the center of the structure with a mass equal to the mass of the whole structure.

(2) The total acting force of all structure bodies on a body inside a ring structure is equal to zero.

These two principles have been generalized to the case of continuous mass distribution over a spherical layer in [10]. One can show that they are valid for a continuous plane ring layer as well. They are locally violated for a ring composed of discrete bodies; however, these local violations are significantly smoothed over in orbital body motion and, hence, the suggested structure can exist.

Let us consider a multilayer K-ring structure; each ring with the number k contains n_k bodies with equal masses ($m_{k,ik} = m_k$, where the index $ik = i_k = 1, 2, \ldots, n_k$ is the number of a body on the k-th ring). According to principle 2, each body on the ring k is not affected by the bodies on external rings with the numbers from $k+1$ to K. Again, according to principle 1 and Eq. (1), the action force on each body of m_k in mass can be written as

$$\vec{F}_k = -\frac{Gm_k\vec{r}_k}{r_k^3}\left[m_0 + \sum_{j=1}^{k-1} n_j\cdot m_j + m_k\cdot f(n_k)\right]. \tag{8}$$

Then, according to the notations for Eq. (3), the interaction parameter for each body on the k-th ring is

$$\mu_{1k} = -G\left[m_0 + \sum_{j=1}^{k-1} n_jm_j + m_kf(n_k)\right], \tag{9}$$

and the trajectory parameter of this body is

$$\alpha_{1k} = \frac{\mu_{1k}}{R_{pk}\cdot x_{pk}^2}. \tag{10}$$

Let $\varphi_{0,k,l}$ designate the angular position of the first body $B_{k,l}$ on the k-th ring at the initial time $t = 0$. Then the polar angles of all bodies on the k-th ring is defined as

$$\varphi_{0,k,ik} = \varphi_{0,k,l} + (i_k - 1)\Delta\varphi_k, \tag{11}$$

where $i_k = 1, \ldots, n_k$ and $\Delta\varphi_k = 2\pi/n_k$.

All bodies on the k-th ring have the same polar radius $r_{0,k}$ at the initial time; therefore, the polar angle can be found from Eq. (3) by the preset angle $\varphi_{0,k,l}$ of the first body and the pericenter radius $R_{p,k}$. All bodies on the k-th ring have an equal pericenter radius and trajectory parameter $\alpha_{1,k}$; the Cartesian coordinates $x_{0,k,ik}$ and $y_{0,k,ik}$ and the velocities $v_{x0,k,ik}$ and $v_{y0,k,ik}$ are determined from Eqs. (6)–(7) with the found radius $r_{0,k}$ and the polar angle $\varphi_{0,k,ik}$ of each body. The radial $v_{r0,k}$ and transversal $v_{t0,k}$ velocities in Eqs. (6)–(7) are equal for all bodies on the k-th ring and defined by Eqs. (4).

The input parameters of a ring structure are the number of rings K, the central body mass m_0, and six parameters of each ring: the number of bodies n_k on the ring, the peripheral body mass m_k, the pericenter radius $R_{p,k}$, the eccentricity of orbit e_k or the trajectory parameter $\alpha_{1,k}$, and the initial angle of the first particle $\varphi_{0,k,l}$ and the angle of its pericenter $\varphi_{p,k}$, i.e., $P = 6K+2$ parameters in total. Using these parameters, the interaction parameter μ_{1k} is found from Eq. (9) and the velocity in pericenters $v_{p,k}$ is found from Eq. (10). Then, using the above-described algorithm, we find the coordinates and velocities of all bodies of the ring structure.

Knowing free parameters of a ring structure, one may establish different additional conditions for producing certain properties of the structure. For example, axisymmetric structures, in which bodies orbit with the same angular velocity, are considered in [4, 7]. Such a structure can be defined by the conditions $\alpha_{1,k} = -1$ and $T_k = T$. $2\cdot K$ parameters of all possible P ones are to be used to write a set of $2\cdot K$ algebraic equations according to Eqs. (5), and the parameters will be found from the solution of this set of equations.

We considered two groups of ring structures, a mass of bodies which was equal to the mass m_{SS} of the solar system. In the first group, the central body mass was equal to the solar mass $m_0 = m_S$, and, in the second group, $m_0 = 0.5m_S$. The structures consisted of three rings, i.e., $K = 3$, and the total number of bodies $N =$

Parameters of six ring structure models: peripheral body masses m_k, in the peripheral body mass on the first ring; pericenter radii $R_{p,\,k}$ in astronomical units (a.u.); and orbital periods T in years. Dn is the dynamics characteristics over 30 years. "Stable" means the absence of visible changes in over 30 years and "US" means unstable; "US, 23 y" for the 10th model means that the structure ruptures in 23 years

Parameters	Values of the parameters for each ring in ring-structure models 1, 2, and 3								
	Mod. 1, $m_0 = M_s$			Mod. 2, $m_0 = M_s$			Mod. 3, $m_0 = 0.5M_s$		
	1	*2*	*3*	*1*	*2*	*3*	*1*	*2*	*3*
n_k	5	7	8	5	7	8	5	7	8
m_k	1	2	3	1	2	3	1	2	3
$R_{p,\,k}$	1	2	3	1	2	3	1	2	3
e_k	0	0	0	0.05	0.1	0.15	0.2	0.05	0.1
T_k	1.4	4	7.3	1.4	4.3	8.7	1.4	2.8	4.8
Dn	Stable			Stable			US		
Parameters	Values of the parameters for each ring in ring-structure models 4, 5, and 10								
	Mod. 4, $m_0 = 0.5M_s$			Mod. 5, $m_0 = 0.5M_s$			Mod. 10, $m_0 = 0.5M_s$		
	1	*2*	*3*	*1*	*2*	*3*	*1*	*2*	*3*
n_k	5	7	8	5	7	8	5	7	8
m_k	1	2	3	1	2	3	1	2	3
$R_{p,\,k}$	3	4.5	6	1	2	3	1	10	16
e_k	0	0	0	0	0	0	0	0	0
T_k	7.3	12.2	26.2	1.4	3.4	5.6	1.4	39.9	70.3
Dn	US			US			US, 23 y		

21. All six types of parameter varied. The differential motion equations were numerically integrated over the precalculated initial conditions by the method described in [11]; the body motion in ring structures was considered. The table presents the main parameters of six models and the conclusion on stability. The angular parameters $\varphi_{0,\,k,\,l}$ and $\varphi_{p,\,k}$ insignificantly affect these models; therefore, they are absent in the table.

The orbits were circular in model 1 from the first group of models and elliptical in model 2 with ring eccentricities of 0.05, 0.1, and 0.15. Figure 2 presents the starting and ending (after 30 years) body positions

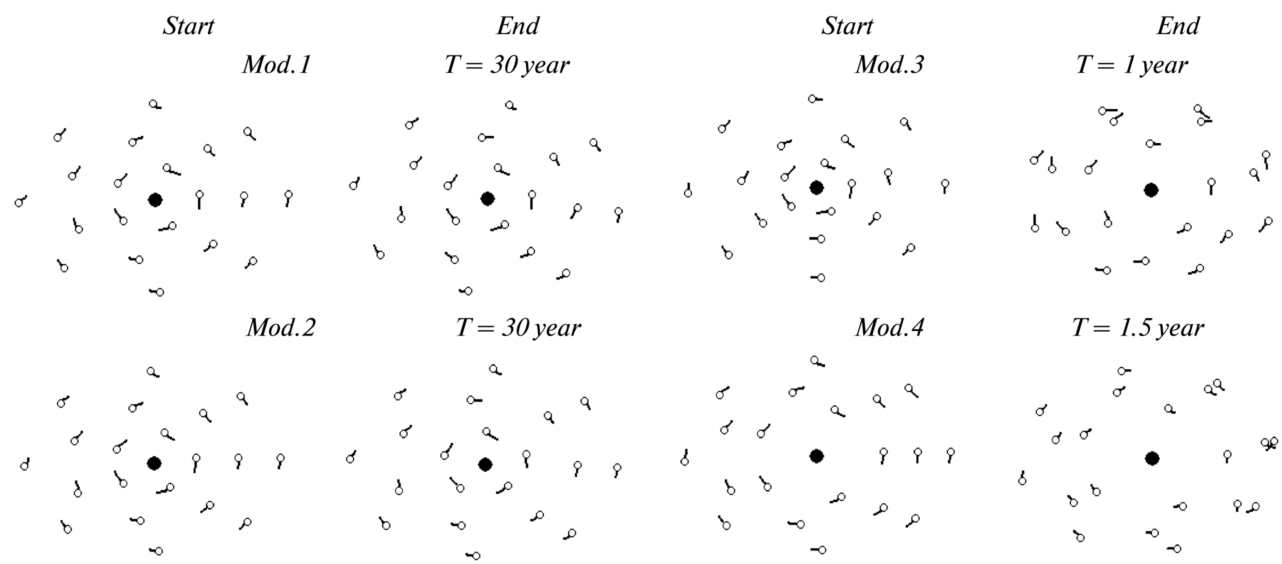

Fig. 2. Dynamics of ring structure models 1, 2, 3, and 4. The central body mass $m_0 = m_S$ in models 1 and 2 and $m_0 = 0.5m_S$ in models 3 and 4. Variations of the parameters e_k, n_k, m_k, and $R_{p,\,k}$ are given in the table. The central body is black; body sizes do not correspond to masses; lines near the bodies show the directions and values of their velocities.

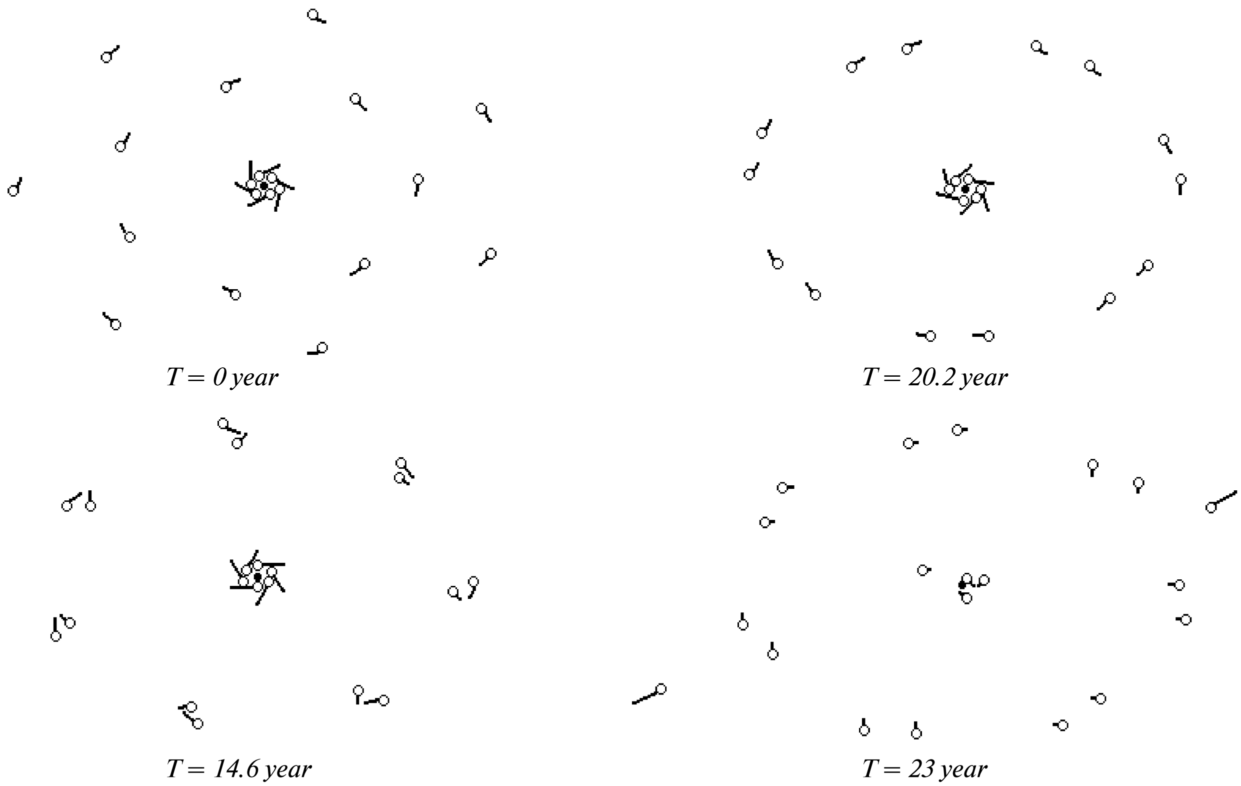

Fig. 3. Dynamics of the 10th ring structure model, $m_0 = 0.5m_S$. Explanations can be found in Fig. 2.

in these models. As can be seen from the table, the orbital time of bodies in rings varies from 1.4 to 8.7 years. No visible changes are observed in these models for 30 years of motion. Therefore, they can be considered stable. In the second group of models with large peripheral body masses, the pericenters of orbits were consequently increased from 3 to 16 a.u. Changes in these models appear at the first stage (see models 3 and 4 in Fig. 2); then they become stronger and the ring structure ruptures. Hence, the models are unstable. The instability is caused by the local interaction of peripheral bodies.

The dynamics of the 10th model is considered in Fig. 3 in more detail. In the beginning, bodies from the first ring move without visible ruptures of the axisymmetric structure, while the bodies from the second and third rings tend to approach and, as is seen, form a common ring in 20.2 years. Then the internal ring ruptures. Two of its bodies are thrown out of the ring structure in opposite directions with high velocities. As is seen from the length of lines, their velocities exceed the velocities of other bodies several times.

CONCLUSIONS

(1) A multilayer ring structure algorithm has been suggested.

(2) The dynamics of ten models has been studied by methods of the numerical integration of differential equations.

(3) Several stable and unstable models have been defined and studied.

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